Fundamentals of Compressible Fluid Mechanics

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‘We are like dwarfs sitting on the shoulders of giants”

from The Metalogicon by John in 1159

CONTENTS

LIST OF FIGURES

LIST OF TABLES

NOMENCLATURE

$\bar{R}$	Universal gas constant, see equation (2.26), page 29
$\boldsymbol{\tau}$	The shear stress Tenser, see equation (3.33), page 52
ℓ	Units length., see equation (2.1), page 25
$\mathfrak{M}$	Angular Momentum, see equation (3.43), page 54
$\boldsymbol{F}_{ext}$	External forces by non–fluids means, see equation (3.36), page 53
ρ	Density of the fluid, see equation (4.1), page 68
B	bulk modulus, see equation (4.38), page 76
B_f	Body force, see equation (2.9), page 27
c	Speed of sound, see equation (4.1), page 68
C_p	Specific pressure heat, see equation (2.23), page 29
C_v	Specific volume heat, see equation (2.22), page 29
E	Young's modulus, see equation (4.50), page 81
E_U	Internal energy, see equation (2.3), page 26
E_u	Internal Energy per unit mass, see equation (2.6), page 26
E_i	System energy at state i, see equation (2.2), page 26
H	Enthalpy, see equation (2.18), page 28
h	Specific enthalpy, see equation (2.18), page 28

k	the ratio of the specific heats, see equation (2.24), page 29
k_T	Fluid thermal conductivity, see equation (3.47), page 56
M	Mach number, see equation (5.8), page 86
n	The polytropic coefficient, see equation (4.35), page 75
P	Pressure, see equation (4.3), page 68
q	Energy per unit mass, see equation (2.6), page 26
Q_{12}	The energy transferred to the system between state 1 and state 2, see equation (2.2), page 26
R	Specific gas constant, see equation (2.27), page 30
R_{mix}	The universal gas constant for mixture, see equation (4.61), page 83
S	Entropy of the system, see equation (2.13), page 28
t	Time, see equation (4.18), page 71
U	velocity , see equation (2.4), page 26
w	Work per unit mass, see equation (2.6), page 26
W_{12}	The work done by the system between state 1 and state 2, see equation (2.2), page 26
z	The compressibility factor, see equation (4.22), page 72

The Book Change Log

Version 0.4.9.8 (second beautified version)

On 17th March 2013 (3.7M pp. 465)

- Real gas issues correction by Guy de Carufel - Canada
- Continue enhance the figures in various chapters.
- Add short dimensional analysis for sound chapter.
- Other minor corrections.

Version 0.4.9.7 (second beautified version)

On 26th Feb 2013 (3.7M pp. 465)

- Real gas issues in isentropic flow and filling chambers
- Continue enhance the figures in various chapters.
- Add short appendix for oblique shock history.
- Enhance the indexes of the book.
- Other minor corrections.

Version 0.4.9.6 (second beautified version)

On 14th Feb 2013 (3.7M pp. 462)

- Provide additional algorithm for fld and the pressure ratio Fanno Flow.

- Continue enhance the figures in various chapters.
- Other minor corrections.

Version 0.4.9.5 (beautified version)

On 20th Jan 2013 (3.7M pp. 458)

- Enhance the discussion and examples on the nozzle efficiency.
- Enhance the figures in various chapters.

Version 0.4.9.4

On 24th Dec 2012 (3.7M pp. 452)

- Enhance the discussion and examples on the maximum temperature in shock tube.

Version 0.4.9.3

On 11th Dec 2012 (3.7M pp. 442)

- Add two examples to the Isentropic chapter.
- Fixing the format and graphics in Isentropic chapter.

Version 0.4.9.2

On 10th Dec 2012 (3.7M pp. 440)

- Add the section on speed of sound in variable density liquid due the gravity with constant gravity.
- English and typo corrections.

Version 0.4.9.1

On 29th Nov 2012 (3.6M pp. 438)

- Maximum work that can be obtained from a cylinder piston configuration
- English and typos corrections.

Version 0.4.9.0

On 13rd Feb 2012 (3.6M pp. 432)

- Significant Enhancement the shock tube section.

- Update the book to compile with the current potto.sty.
- insert the introduction to fluid mechanics.
- English and typo corrections.

Version 0.4.8.8

On 29th Dec 2011 (3.6M pp. 386)

- Add two figures explain the maximum Mach number limits in the shock tube.
- English and typo corrections.

Version 0.4.8.7

On 29th Dec 2011 (3.6M pp. 386)

- Significantly improved the shock tube section.
- Improvements of the structure to meed to the standard.
- English and typo corrections.

Version 0.4.8.6

On 23rd Oct 2009 (3.6M pp. 384)

- Add the section about Theodor Meyer's biography
- Addition of Temperature Velocity diagram. (The addition to the other chapters was not added yet).

Version 0.4.8.5b

On 07th Sep 2009 (3.5M pp. 376)

- Corrections in the Fanno chapter in Trends section.
- English corrections.

Version 0.4.8.5a

On 04th July 2009 (3.5M pp. 376)

- Corrections in the thermodynamics chapter to the gases properties table.
- English corrections.
- Improve the multilayer sound traveling example (Heru's suggestion)

Version 0.4.8.5a

On 04th July 2009 (3.3M pp. 380)

- Correction to the gases properties table (Michael Madden and Heru Reksoprodjo)
- English corrections.
- Improving the multilayer sound wave traveling

Version 0.4.8.5

On 14th January 2009 (3.3M pp. 380)

- Improve images macro (two captions issue).
- English corrections.

Version 0.4.8.5rc

On 31st December 2008 (3.3M pp. 380)

- Add Gary Settles's color image in wedge shock and an example.
- Improve the wrap figure issue to oblique shock.
- Add Moody diagram to Fanno flow.
- English corrections to the oblique shock chapter.

Version 0.4.8.4

On 7th October 2008 (3.2M pp. 376)

- More work on the nomenclature issue.
- Important equations and useful equations issues inserted.
- Expand the discussion on the friction factor in isothermal and fanno flow.

Version 0.4.8.3

On 17th September 2008 (3.1M pp. 369)

- Started the nomenclature issue so far only the thermodynamics chapter.
- Started the important equations and useful equations issue.
- Add the introduction to thermodynamics chapter.
- Add the discussion on the friction factor in isothermal and fanno flow.

Version 0.4.8.2

On 25th January 2008 (3.1M pp. 353)

- Add several additions to the isentropic flow, normal shock,
- Rayleigh Flow.
- Improve some examples.
- More changes to the script to generate separate chapters sections.
- Add new macros to work better so that php and pdf version will be similar.
- More English revisions.

Version 0.4.8

November-05-2007

- Add the new unchoked subsonic Fanno Flow section which include the "unknown" diameter question.
- Shock (Wave) drag explanation with example.
- Some examples were add and fixing other examples (small perturbations of oblique shock).
- Minor English revisions.

Version 0.4.4.3pr1

July-10-2007

- Improvement of the pdf version provide links.

Version 0.4.4.2a

July-4-2007 version

- Major English revisions in Rayleigh Flow Chapter.
- Continue the improvement of the HTML version (imageonly issues).
- Minor content changes and addition of an example.

Version 0.4.4.2

May-22-2007 version

- Major English revisions.
- Continue the improvement of the HTML version.
- Minor content change and addition of an example.

Version 0.4.4.1

Feb-21-2007 version

- Include the indexes subjects and authors.
- Continue the improve the HTML version.
- solve problems with some of the figures location (float problems)
- Improve some spelling and grammar.
- Minor content change and addition of an example.
- The main change is the inclusion of the indexes (subject and authors). There were some additions to the content which include an example. The "naughty professor's questions" section isn't completed and is waiting for interface of Potto-GDC to be finished (engine is finished, hopefully next two weeks). Some grammar and misspelling corrections were added.

  Now include a script that append a title page to every pdf fraction of the book (it was fun to solve this one). Continue to insert the changes (log) to every source file (latex) of the book when applicable. This change allows to follow the progression of the book. Most the tables now have the double formatting one for the html and one for the hard copies.

Version 0.4.4pr1

Jan-16-2007 version

- Major modifications of the source to improve the HTML version.
- Add the naughty professor's questions in the isentropic chapter.
- Some grammar and miss spelling corrections.

Version 0.4.3.2rc1

Dec-04-2006 version

- Add new algorithm for Fanno Flow calculation of the shock location in the supersonic flow for given fld (exceeding Max) and M1 (see the example).
- Minor addition in the Sound and History chapters.
- Add analytical expression for Mach number results of piston movement.

Version 0.4.3.1rc4 aka 0.4.3.1

Nov-10-2006 aka Roy Tate's version

For this release (the vast majority) of the grammatical corrections are due to Roy Tate

- Grammatical corrections through the history chapter and part of the sound chapter.
- Very minor addition in the Isothermal chapter about supersonic branch.

Version 0.4.3.1rc3

Oct-30-2006

- Add the solutions to last three examples in Chapter Normal Shock in variable area.
- Improve the discussion about partial open and close moving shock dynamics i.e. high speed running into slower velocity
- Clean other tables and figure and layout.

Version 0.4.3rc2

Oct-20-2006

- Clean up of the isentropic and sound chapters
- Add discussion about partial open and close moving shock dynamics i.e. high speed running into slower velocity.
- Add the partial moving shock figures (never published before)

Version 0.4.3rc1

Sep-20-2006

- Change the book's format to 6x9 from letter paper
- Clean up of the isentropic chapter.
- Add the shock tube section
- Generalize the discussion of the the moving shock (not including the change in the specific heat (material))
- Add the Impulse Function for Isothermal Nozzle section
- Improve the discussion of the Fliegner's equation
- Add the moving shock table (never published before)

Version 0.4.1.9 (aka 0.4.1.9rc2)

May-22-2006

- Added the Impulse Function
- Add two examples.
- Clean some discussions issues .

Version 0.4.1.9rc1

May-17-2006

- Added mathematical description of Prandtl-Meyer's Function
- Fixed several examples in oblique shock chapter
- Add three examples.
- Clean some discussions issues .

Version 0.4.1.8 aka Version 0.4.1.8rc3

May-03-2006

- Added Chapman's function
- Fixed several examples in oblique shock chapter
- Add two examples.
- Clean some discussions issues .

Version 0.4.1.8rc2

Apr-11-2006

- Added the Maximum Deflection Mach number's equation
- Added several examples to oblique shock

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- **Date(s) of contribution(s):** 2004 to present
- **Nature of contribution:** Original author.
- **Contact at:** `barmeir at gmail.com`

John Martones

- **Date(s) of contribution(s):** June 2005

- **Nature of contribution:** HTML formatting, some error corrections.

Grigory Toker

- **Date(s) of contribution(s):** August 2005
- **Nature of contribution:** Provided pictures of the oblique shock for oblique shock chapter.

Ralph Menikoff

- **Date(s) of contribution(s):** July 2005
- **Nature of contribution:** Some discussions about the solution to oblique shock and about the Maximum Deflection of the oblique shock.

Domitien Rataaforret

- **Date(s) of contribution(s):** Oct 2006
- **Nature of contribution:** Some discussions about the French problem and help with the new wrapImg command.

Gary Settles

- **Date(s) of contribution(s):** Dec 2008, July 2009
- **Nature of contribution:** Four images for oblique shock two dimensional, and cone flow.
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Typo corrections and other "minor" contributions

- **H. Gohrah, Ph. D.**, September 2005, some LaTeX issues.
- **Roy Tate** November 2006, Suggestions on improving English and grammar.
- **Nancy Cohen** 2006, Suggestions on improving English and style for various issues.
- **Irene Tan** 2006, proof reading many chapters and for various other issues.
- **Michael Madden** 2009, gas properties table corrections
- **Heru Reksoprodjo** 2009, point to affecting dimensional parameter in multi layer sound travel, and also point to the mistake in the gas properties.
- **Raghvendra Gupta, Ph. D.**, Nov 2013, Correction of example 5.1.

About This Author

Genick Bar-Meir holds a Ph.D. in Mechanical Engineering from University of Minnesota and a Master in Fluid Mechanics from Tel Aviv University. Dr. Bar-Meir was the last student of the late Dr. R.G.E. Eckert. Much of his time has been spend doing research in the field of heat and mass transfer (related to renewal energy issues) and this includes fluid mechanics related to manufacturing processes and design. Currently, he spends time writing books (there are already three very popular books) and softwares for the POTTO project (see Potto Prologue). The author enjoys to encourage his students to understand the material beyond the basic requirements of exams.

In his early part of his professional life, Bar-Meir was mainly interested in elegant models whether they have or not a practical applicability. Now, this author's views had changed and the virtue of the practical part of any model becomes the essential part of his ideas, books and software.

He developed models for Mass Transfer in high concentration that became a building blocks for many other models. These models are based on analytical solution to a family of equations[1]. As the change in the view occurred, Bar-Meir developed models that explained several manufacturing processes such the rapid evacuation of gas from containers, the critical piston velocity in a partially filled chamber (related to hydraulic jump), application of supply and demand to rapid change power system and etc. All the models have practical applicability. These models have been extended by several research groups (needless to say with large research grants). For example, the Spanish Comision Interministerial provides grants TAP97-0489 and PB98-0007, and the CICYT and the European Commission provides 1FD97-2333 grants for minor aspects of that models. Moreover, the author's models were used in numerical works, in GM, British industry, Spain, and Canada.

In the area of compressible flow, it was commonly believed and taught that there is only weak and strong shock and it is continue by Prandtl–Meyer function. Bar–

[1]Where the mathematicians were able only to prove that the solution exists.

Meir discovered the analytical solution for oblique shock and showed that there is a quiet buffer between the oblique shock and Prandtl–Meyer. He also build analytical solution to several moving shock cases. He described and categorized the filling and evacuating of chamber by compressible fluid in which he also found analytical solutions to cases where the working fluid was ideal gas. The common explanation to Prandtl–Meyer function shows that flow can turn in a sharp corner. Engineers have constructed design that based on this conclusion. Bar-Meir demonstrated that common Prandtl–Meyer explanation violates the conservation of mass and therefor the turn must be around a finite radius. The author's explanations on missing diameter and other issues in fanno flow and "naughty professor's question" are used in the industry.

In his book "Basics of Fluid Mechanics", Bar-Meir demonstrated several things which include Pushka equation, dealing with the pressure accounted the slight compressibility (a finite Bulk Modulus effect), speed of sound in slightly compressible liquid. He showed the relationship between the wavy surface and the multi–phases flow.

The author lives with his wife and three children. A past project of his was building a four stories house, practically from scratch. While he writes his programs and does other computer chores, he often feels clueless about computers and programing. While he is known to look like he knows a lot a lot about many things, the author just know to learn quickly. The author spent years working on the sea (ships) as a engine sea officer but now the author prefers to remain on a solid ground.

Prologue For The POTTO Project

This books series was born out of frustrations in two respects. The first issue is the enormous price of college textbooks. It is unacceptable that the price of the college books will be over $150 per book (over 10 hours of work for an average student in The United States).

The second issue that prompted the writing of this book is the fact that we as the public have to deal with a corrupted judicial system. As individuals we have to obey the law, particularly the copyright law with the "infinite[2]" time with the copyright holders. However, when applied to "small" individuals who are not able to hire a large legal firm, judges simply manufacture facts to make the little guy lose and pay for the defense of his work. On one hand, the corrupted court system defends the "big" guys and on the other hand, punishes the small "entrepreneur" who tries to defend his or her work. It has become very clear to the author and founder of the POTTO Project that this situation must be stopped. Hence, the creation of the POTTO Project. As R. Kook, one of this author's sages, said instead of whining about arrogance and incorrectness, one should increase wisdom. This project is to increase wisdom and humility.

The POTTO Project has far greater goals than simply correcting an abusive Judicial system or simply exposing abusive judges. It is apparent that writing textbooks especially for college students as a cooperation, like an open source, is a new idea[3]. Writing a book in the technical field is not the same as writing a novel. The writing of a technical book is really a collection of information and practice. There is always someone who can add to the book. The study of technical material isn't only done by having to memorize the material, but also by coming to understand and be able to solve

[2]After the last decision of the Supreme Court in the case of Eldred v. Ashcroff (see `http://cyber.law.harvard.edu/openlaw/eldredvashcroft` for more information) copyrights practically remain indefinitely with the holder (not the creator).

[3]In some sense one can view the encyclopedia Wikipedia as an open content project (see `http://en.wikipedia.org/wiki/Main_Page`). The wikipedia is an excellent collection of articles which are written by various individuals.

related problems. The author has not found any technique that is more useful for this purpose than practicing the solving of problems and exercises. One can be successful when one solves as many problems as possible. To reach this possibility the collective book idea was created/adapted. While one can be as creative as possible, there are always others who can see new aspects of or add to the material. The collective material is much richer than any single person can create by himself.

The following example explains this point: The army ant is a kind of carnivorous ant that lives and hunts in the tropics, hunting animals that are even up to a hundred kilograms in weight. The secret of the ants' power lies in their collective intelligence. While a single ant is not intelligent enough to attack and hunt large prey, the collective power of their networking creates an extremely powerful intelligence to carry out this attack[4]. When an insect which is blind can be so powerful by networking, So can we in creating textbooks by this powerful tool.

Why would someone volunteer to be an author or organizer of such a book? This is the first question the undersigned was asked. The answer varies from individual to individual. It is hoped that because of the open nature of these books, they will become the most popular books and the most read books in their respected field. For example, the books on compressible flow and die casting became the most popular books in their respective area. In a way, the popularity of the books should be one of the incentives for potential contributors. The desire to be an author of a well–known book (at least in his/her profession) will convince some to put forth the effort. For some authors, the reason is the pure fun of writing and organizing educational material. Experience has shown that in explaining to others any given subject, one also begins to better understand the material. Thus, contributing to these books will help one to understand the material better. For others, the writing of or contributing to this kind of books will serve as a social function. The social function can have at least two components. One component is to come to know and socialize with many in the profession. For others the social part is as simple as a desire to reduce the price of college textbooks, especially for family members or relatives and those students lacking funds. For some contributors/authors, in the course of their teaching they have found that the textbook they were using contains sections that can be improved or that are not as good as their own notes. In these cases, they now have an opportunity to put their notes to use for others. Whatever the reasons, the undersigned believes that personal intentions are appropriate and are the author's/organizer's private affair.

If a contributor of a section in such a book can be easily identified, then that contributor will be the copyright holder of that specific section (even within question/answer sections). The book's contributor's names could be written by their sections. It is not just for experts to contribute, but also students who happened to be doing their homework. The student's contributions can be done by adding a question and perhaps the solution. Thus, this method is expected to accelerate the creation of these high quality books.

These books are written in a similar manner to the open source software

[4]see also in Franks, Nigel R.; "Army Ants: A Collective Intelligence," American Scientist, 77:139, 1989 (see for information `http://www.ex.ac.uk/bugclub/raiders.html`)

process. Someone has to write the skeleton and hopefully others will add "flesh and skin." In this process, chapters or sections can be added after the skeleton has been written. It is also hoped that others will contribute to the question and answer sections in the book. But more than that, other books contain data[5] which can be typeset in LaTeX. These data (tables, graphs and etc.) can be redone by anyone who has the time to do it. Thus, the contributions to books can be done by many who are not experts. Additionally, contributions can be made from any part of the world by those who wish to translate the book.

It is hoped that the books will be error-free. Nevertheless, some errors are possible and expected. Even if not complete, better discussions or better explanations are all welcome to these books. These books are intended to be "continuous" in the sense that there will be someone who will maintain and improve the books with time (the organizer(s)).

These books should be considered more as a project than to fit the traditional definition of "plain" books. Thus, the traditional role of author will be replaced by an organizer who will be the one to compile the book. The organizer of the book in some instances will be the main author of the work, while in other cases only the gate keeper. This may merely be the person who decides what will go into the book and what will not (gate keeper). Unlike a regular book, these works will have a version number because they are alive and continuously evolving.

The undersigned of this document intends to be the organizer–author–coordinator of the projects in the following areas:

Table -1. Books under development in Potto project.

Project Name	Progress	Remarks	Version	Availability for Public Download	Number DownLoads
Compressible Flow	beta		0.4.8.4	✔	120,000
Die Casting	alpha		0.1	✔	60,000
Dynamics	NSY		0.0.0	✖	-
Fluid Mechanics	alpha		0.1.8	✔	15,000
Heat Transfer	NSY	Based on Eckert	0.0.0	✖	-
Mechanics	NSY		0.0.0	✖	-
Open Channel Flow	NSY		0.0.0	✖	-
Statics	early alpha	first chapter	0.0.1	✖	-
Strength of Material	NSY		0.0.0	✖	-

[5] Data are not copyrighted.

Table -1. Books under development in Potto project. (continue)

Project Name	Progress	Remarks	Version	Availability for Public Download	Number DownLoads
Thermodynamics	early alpha		0.0.01	✖	-
Two/Multi phases flow	NSY	Tel-Aviv'notes	0.0.0	✖	-

NSY = Not Started Yet

The meaning of the progress is as:

- The Alpha Stage is when some of the chapters are already in a rough draft;
- in Beta Stage is when all or almost all of the chapters have been written and are at least in a draft stage;
- in Gamma Stage is when all the chapters are written and some of the chapters are in a mature form; and
- the Advanced Stage is when all of the basic material is written and all that is left are aspects that are active, advanced topics, and special cases.

The mature stage of a chapter is when all or nearly all the sections are in a mature stage and have a mature bibliography as well as numerous examples for every section. The mature stage of a section is when all of the topics in the section are written, and all of the examples and data (tables, figures, etc.) are already presented. While some terms are defined in a relatively clear fashion, other definitions give merely a hint on the status. But such a thing is hard to define and should be enough for this stage.

The idea that a book can be created as a project has mushroomed from the open source software concept, but it has roots in the way science progresses. However, traditionally books have been improved by the same author(s), a process in which books have a new version every a few years. There are book(s) that have continued after their author passed away, i.e., the *Boundary Layer Theory* originated[6] by Hermann Schlichting but continues to this day. However, projects such as the Linux Documentation project demonstrated that books can be written as the cooperative effort of many individuals, many of whom volunteered to help.

Writing a textbook is comprised of many aspects, which include the actual writing of the text, writing examples, creating diagrams and figures, and writing the

[6]Originally authored by Dr. Schlichting, who passed way some years ago. A new version is created every several years.

LaTeX macros[7] which will put the text into an attractive format. These chores can be done independently from each other and by more than one individual. Again, because of the open nature of this project, pieces of material and data can be used by different books.

[7] One can only expect that open source and readable format will be used for this project. But more than that, only LaTeX, and perhaps troff, have the ability to produce the quality that one expects for these writings. The text processes, especially LaTeX, are the only ones which have a cross platform ability to produce macros and a uniform feel and quality. Word processors, such as OpenOffice, Abiword, and Microsoft Word software, are not appropriate for these projects. Further, any text that is produced by Microsoft and kept in "Microsoft" format are against the spirit of this project In that they force spending money on Microsoft software.

Prologue For This Book

Version 0.4.9 pp. ? Feb ?, 2012

over 400,000 downloads

In the last three years the focus was on building the fluid mechanics book. In the construction of the fluid book the potto style file significantly changed to the the point that render the old files of book as un–compilable. This work was to bring these file up to date. Several chapters from that the fluid book were summarized into single introduction chapter on Fluid Mechanics. There are several additions which include better description of the shock tube, and sound in variable liquid density etc.

Version 0.4.8.5a . July 21, 2009

over 150,000 downloads

The spread of the book was the biggest change that can be observed during the last year (more than a year). Number of download reached to over 160,000 copies. The book became the main textbook in many universities. This time, the main work focused on corrections and minor additions. The fluid mechanics book is under construction and reached to 0.17x version. Hopefully when finished, with good help in the coming months will be used in this book to make better introduction. Other material in this book like the gas dynamics table and equation found their life and very popular today. This additions also include GDC which become the standard calculator for the gas dynamics class.

Version 0.4.8 Jan. 23, 2008

It is more than a year ago, when the previous this section was modified. Many things have changed, and more people got involved. It nice to know that over 70,000 copies have been download from over 130 countries. It is more pleasant to find that this book is used in many universities around the world, also in many institutes like NASA (a tip from Dr. Farassat, NASA "to educate their "young scientist, and engineers") and others. Looking back, it must be realized that while, this book is the best in many areas, like oblique shock, moving shock, fanno flow, etc there are missing some sections, like methods of characteristics, and the introductory sections (fluid mechanics, and thermodynamics). Potto–GDC is much more mature and it is changing from "advance look up" to a real gas dynamics calculator (for example, calculation of unchoked Fanno Flow). Today Potto–GDC has the only capability to produce the oblique shock figure. Potto-GDC is becoming the major educational educational tool in gas dynamics. To kill two birds in one stone, one, continuous requests from many and, two, fill the introductory section on fluid mechanics in this book this area is major efforts in the next few months for creating the version 0.2 of the "Basic of Fluid Mechanics" are underway.

Version 0.4.3 Sep. 15, 2006

The title of this section is change to reflect that it moved to beginning of the book. While it moves earlier but the name was not changed. Dr. Menikoff pointed to this inconsistency, and the author is apologizing for this omission.

Several sections were add to this book with many new ideas for example on the moving shock tables. However, this author cannot add all the things that he was asked and want to the book in instant fashion. For example, one of the reader ask why not one of the example of oblique shock was not turn into the explanation of von Neumann paradox. The author was asked by a former client why he didn't insert his improved tank filling and evacuating models (the addition of the energy equation instead of isentropic model). While all these requests are important, the time is limited and they will be inserted as time permitted.

The moving shock issues are not completed and more work is needed also in the shock tube. Nevertheless, the ideas of moving shock will reduced the work for many student of compressible flow. For example solving homework problem from other text books became either just two mouse clicks away or just looking at that the tables in this book. I also got request from a India to write the interface for Microsoft. I am sorry will not be entertaining work for non Linux/Unix systems, especially for Microsoft. If one want to use the software engine it is okay and permitted by the license of this work.

The download to this mount is over 25,000.

Version 0.4.2

It was surprising to find that over 14,000 downloaded and is encouraging to receive over 200 thank you eMail (only one from U.S.A./Arizona) and some other reactions. This textbook has sections which are cutting edge research[8].

The additions of this version focus mainly on the oblique shock and related issues as results of questions and reactions on this topic. However, most readers reached to www.potto.org by searching for either terms "Rayleigh flow" (107) and "Fanno flow" ((93). If the total combined variation search of terms "Fanno" and "Rayleigh" (mostly through google) is accounted, it reaches to about 30% (2011). This indicates that these topics are highly is demanded and not many concerned with the shock phenomena as this author believed and expected. Thus, most additions of the next version will be concentrated on Fanno flow and Rayleigh flow. The only exception is the addition to Taylor–Maccoll flow (axisymmetricale conical flow) in Prandtl–Meyer function (currently in a note form).

Furthermore, the questions that appear on the net will guide this author on what is really need to be in a compressible flow book. At this time, several questions were about compressibility factor and two phase flow in Fanno flow and other kind of flow models. The other questions that appeared related two phase and connecting several chambers to each other. Also, an individual asked whether this author intended to write about the unsteady section, and hopefully it will be near future.

Version 0.4

Since the last version (0.3) several individuals sent me remarks and suggestions. In the introductory chapter, extensive description of the compressible flow history was written. In the chapter on speed of sound, the two phase aspects were added. The isothermal nozzle was combined with the isentropic chapter. Some examples were added to the normal shock chapter. The fifth chapter deals now with normal shock in variable area ducts. The sixth chapter deals with external forces fields. The chapter about oblique shock was added and it contains the analytical solution. At this stage, the connection between Prandtl–Meyer flow and oblique is an note form. The a brief chapter on Prandtl–Meyer flow was added.

Version 0.3

In the traditional class of compressible flow it is assumed that the students will be aerospace engineers or dealing mostly with construction of airplanes and turbomachinery. This premise should not be assumed. This assumption drives students from other fields away from this knowledge. This knowledge should be spread to other fields because it needed there as well. This "rejection" is especially true when students feel that they have to go through a "shock wave" in their understanding.

[8] A reader asked this author to examine a paper on Triple Shock Entropy Theorem and Its Consequences by Le Roy F. Henderson and Ralph Menikoff. This led to comparison between maximum to ideal gas model to more general model.

This book is the second book in the series of POTTO project books. POTTO project books are open content textbooks. The reason the topic of Compressible Flow was chosen, while relatively simple topics like fundamentals of strength of material were delayed, is because of the realization that manufacture engineering simply lacks fundamental knowledge in this area and thus produces faulty designs and understanding of major processes. Unfortunately, the undersigned observed that many researchers who are dealing with manufacturing processes are lack of understanding about fluid mechanics in general but particularly in relationship to compressible flow. In fact one of the reasons that many manufacturing jobs are moving to other countries is because of the lack of understanding of fluid mechanics in general and compressible in particular. For example, the lack of competitive advantage moves many of the die casting operations to off shore[9]. It is clear that an understanding of Compressible Flow is very important for areas that traditionally have ignored the knowledge of this topic[10].

As many instructors can recall from their time as undergraduates, there were classes during which most students had a period of confusion, and then later, when the dust settled, almost suddenly things became clear. This situation is typical also for Compressible Flow classes, especially for external compressible flow (e.g. flow around a wing, etc.). This book offers a more balanced emphasis which focuses more on internal compressible flow than the traditional classes. The internal flow topics seem to be common for the "traditional" students and students from other fields, e.g., manufacturing engineering.

This book is written in the spirit of my adviser and mentor E.R.G. Eckert. Who, aside from his research activity, wrote the book that brought a revolution in the heat transfer field of education. Up to Eckert's book, the study of heat transfer was without any dimensional analysis. He wrote his book because he realized that the dimensional analysis utilized by him and his adviser (for the post doc), Ernst Schmidt, and their colleagues, must be taught in engineering classes. His book met strong criticism in which some called to burn his book. Today, however, there is no known place in world that does not teach according to Eckert's doctrine. It is assumed that the same kind of individuals who criticized Eckert's work will criticize this work. This criticism will not change the future or the success of the ideas in this work. As a wise person says "don't tell me that it is wrong, show me what is wrong"; this is the only reply. With all the above, it must be emphasized that this book will not revolutionize the field even though considerable new materials that have never been published are included. Instead, it will provide a new emphasis and new angle to Gas Dynamics.

Compressible flow is essentially different from incompressible flow in mainly two respects: discontinuity (shock wave) and choked flow. The other issues, while important, are not that crucial to the understanding of the unique phenomena of compressible flow. These unique issues of compressible flow are to be emphasized and

[9]Please read the undersigned's book "Fundamentals of Die Casting Design," which demonstrates how ridiculous design and research can be.

[10]The fundamental misunderstanding of choking results in poor models (research) in the area of die casting, which in turn results in many bankrupt companies and the movement of the die casting industry to offshore.

shown. Their applicability to real world processes is to be demonstrated[11].

The book is organized into several chapters which, as a traditional textbook, deals with a basic introduction of thermodynamics concepts (under construction). The second chapter deals with speed of sound. The third chapter provides the first example of choked flow (isentropic flow in a variable area). The fourth chapter deals with a simple case of discontinuity (a simple shock wave in a nozzle). The next chapter is dealing with isothermal flow with and without external forces (the moving of the choking point), again under construction. The next three chapters are dealing with three models of choked flow: Isothermal flow[12], Fanno flow and Rayleigh flow. First, the Isothermal flow is introduced because of the relative ease of the analytical treatment. Isothermal flow provides useful tools for the pipe systems design. These chapters are presented almost independently. Every chapter can be "ripped" out and printed independently. The topics of filling and evacuating of gaseous chambers are presented, normally missed from traditional textbooks. There are two advanced topics which included here: oblique shock wave, and properties change effects (ideal gases and real gases) (under construction). In the oblique shock, for the first time analytical solution is presented, which is excellent tool to explain the strong, weak and unrealistic shocks. The chapter on one-dimensional unsteady state, is currently under construction.

The last chapter deals with the computer program, Gas Dynamics Calculator (CDC-POTTO). The program design and how to use the program are described (briefly).

Discussions on the flow around bodies (wing, etc), and Prandtl–Meyer expansion will be included only after the gamma version unless someone will provide discussion(s) (a skeleton) on these topics.

It is hoped that this book will serve the purposes that was envisioned for the book. It is further hoped that others will contribute to this book and find additional use for this book and enclosed software.

[11] If you have better and different examples or presentations you are welcome to submit them.

[12] It is suggested to referred to this model as Shapiro flow

How This Book Was Written

This book started because I needed an explanation for manufacturing engineers. Apparently many manufacturing engineers and even some researchers in manufacturing engineering were lack of understanding about fluid mechanics in particularly about compressible flow. Therefore, I wrote to myself some notes and I converted one of the note to a chapter in my first book, "Fundamentals Of Die Casting Design." Later, I realized that people need down to earth book about compressible flow and this book was born. Later I need a chapter on fluid mechamics introduction so I wrote about fluid mechacnics and several of the chapter from that book were summirized to be included in this book.

The free/open content of the book was created because the realization that open content accelerated the creation of books and reaction to the corruption of the court implementing the copyright law by manufacturing facts and laws. It was farther extended by the allegation of free market and yet the academic education cost is sky rocketing without a real reason and real competition. There is no reason why a textbook which cost at the very most $10 to publish/produce to cost about 150 dollars. If a community will pull together, the best books can be created. Anyone can be part of it. For example, even my 10 years old son, Eliezer made me change the chapter on isothermal flow. He made me realized that the common approach to supersonic branch of isothermal as non–existent is the wrong approach. It should be included because this section provides the explanation and direction on what Fanno flow model will approach if heat transfer is taken into account[13].

I realized that books in compressible flow are written in a form that is hard for non fluid mechanic engineer to understand. Therefore, this book is designed to be in such form that is easy to understand. I wrote notes and asked myself what materials should be included in such a book so when I provide consultation to a company, I do not need to explain the fundamentals. Therefore, there are some chapters in this book

[13]Still in untyped note form.

which are original materials never published before. The presentation of some of the chapters is different from other books. The book does not provide the old style graphical solution methods yet provide the graphical explanation of things.

Of course, this book was written on Linux (MicrosoftLess book). This book was written using the vim editor for editing (sorry never was able to be comfortable with emacs). The graphics were done by TGIF, the best graphic program that this author experienced so far. The old figures were done by grap (part the old Troff). Unfortunately, I did not have any access to grap and switched to Grace. Grace is a problematic program. Finally, the gle is replacing the old grace. So far, it seems much better choice and from version 0.4.8 all will be done using GLE. The spell checking was done by gaspell, a program that cannot be used on a new system and I had to keep my old Linux to make it work[14]. I hope someone will write a new spell check so I can switch to a new system.

The figure in the cover page was created by Michael Petschauer, graphic designer, and is open/free content copyrighted by him (happy_circle@yahoo.com).

[14] If you would like to to help me to write a new spell check user interface, please contact me.

About Gas Dynamics Calculator

Gas Dynamic Calculator, (Potto–GDC) was created to generate various tables for the book either at end the chapters or for the exercises. This calculator was given to several individuals and they found Potto–GDC to be very useful. So, I decided to include Potto–GDC to the book.

Initially, the Potto-GDC was many small programs for specific tasks. For example, the stagnation table was one such program. Later, the code became a new program to find the root of something between the values of the tables e.g. finding parameters for a given $\frac{4fL}{D}$. At that stage, the program changed to contain a primitive interface to provide parameters to carry out the proper calculations. Yet, then, every flow model was a different program.

When it become cumbersome to handle several programs, the author utilized the object oriented feature of C++ and assigned functions to the common tasks to a base class and the specific applications to the derived classes. Later, a need to intermediate stage of tube flow model (the PipeFlow class) was created and new classes were created.

The graphical interface was created only after the engine was written. The graphical interface was written to provide a filter for the unfamiliar user. It also remove the need to recompile the code every time.

Version 0.5

In this version the main point was on the bugs fixing but also add the results can be shown in a HTML code. In fanno flow, many problems of unchoked Fanno flow now possible to solve (by one click).

Version 0.4.3

This version add several features among them is the shock dynamics calculations with the iteration info. The last feature is good for homework either for the students or the instructors.

Version 0.4.1.7

Version 4.1.7 had several bug fixes and add two angle calculations to the oblique shock. Change the logtable to tabular environment for short tables.

Preface

"In the beginning, the POTTO project was without form, and void; and emptiness was upon the face of the bits and files. And the fingers of the Author moved upon the face of the keyboard. And the Author said, Let there be words, and there were words."[15].

This book, Fundamentals of Compressible Flow, describes the fundamentals of compressible flow phenomena for engineers and others. This book is designed to replace the book(s) or instructor's notes for the compressible flow in (mostly) undergraduate classes for engineering/science students. It is hoped that the book could be used as a reference book for people who have at least some knowledge of the basics of fundamental fluid mechanics, and basic science such as calculus, physics, etc. It is hoped that the computer program enclosed in the book will take on a life of its own and develop into an open content or source project.

The structure of this book is such that many of the chapters could be usable independently. For example, if you need information about, say, Fanno flow, you can read just chapter 10. I hope this makes the book easier to use as a reference manual. However, this manuscript is first and foremost a textbook, and secondly a reference manual only as a lucky coincidence.

I have tried to describe why the theories are the way they are, rather than just listing "seven easy steps" for each task. This means that a lot of information is presented which is not necessary for everyone. These explanations have been marked as such and can be skipped.[16] Reading everything will, naturally, increase your understanding of the fundamentals of compressible fluid flow.

This book is written and maintained on a volunteer basis. Like all volunteer work, there is a limit on how much effort I was able to put into the book and its

[15] To the power and glory of the mighty God. This book is only to explain his power.

[16] At the present, the book is not well organized. You have to remember that this book is a work in progress.

organization. Moreover, due to the fact that English is my third language and time limitations, the explanations are not as good as if I had a few years to perfect them. Nevertheless, I believe professionals working in many engineering fields will benefit from this information. This book contains many original models, and explanations never published before.

I have left some issues which have unsatisfactory explanations in the book, marked with a Mata mark. I hope to improve or to add to these areas in the near future. Furthermore, I hope that many others will participate of this project and will contribute to this book (even small contributions such as providing examples or editing mistakes are needed).

I have tried to make this text of the highest quality possible and am interested in your comments and ideas on how to make it better. Incorrect language, errors, ideas for new areas to cover, rewritten sections, more fundamental material, more mathematics (or less mathematics); I am interested in it all. If you want to be involved in the editing, graphic design, or proofreading, please drop me a line. You may contact me via Email at "barmeir@gmail.com".

Naturally, this book contains material that never was published before. This material never went through a peer review. While peer review and publication in a professional publication is excellent idea in theory. In practice, this process leaves a large room to blockage of novel ideas and plagiarism. If you would like be "peer reviews" or critic to my new ideas please send me your idea(s). Even reaction/comments from individuals like David Marshall[17]

Several people have helped me with this book, directly or indirectly. I would like to especially thank to my adviser, Dr. E. R. G. Eckert, whose work was the inspiration for this book. I also would like to thank Amy Ross for her advice ideas, and assistance. Our new volunteer, Irene Tan had done wonderful job.

The symbol META was added to provide typographical conventions to blurb as needed. This is mostly for the author's purposes and also for your amusement. There are also notes in the margin, but those are solely for the author's purposes, ignore them please. They will be removed gradually as the version number advances.

I encourage anyone with a penchant for writing, editing, graphic ability, LaTeX knowledge, and material knowledge and a desire to provide open content textbooks and to improve them to join me in this project. If you have Internet e-mail access, you can contact me at "barmeir@gmail.com".

[17]Dr. Marshall wrote to this author that the author should review other people work before he write any thing new (well, literature review is always good?). Over ten individuals wrote me about this letter. I am asking from everyone to assume that his reaction was innocent one. While his comment looks like unpleasant reaction, it brought or cause the expansion the oblique shock chapter. However, other email that imply that someone will take care of this author aren't appreciated.

To Do List and Road Map

This book is not complete and probably never will be completed. There will always new problems to add or to polish the explanations or include more new materials. Also issues that associated with the book like the software has to be improved. It is hoped the changes in TEX and LATEX related to this book in future will be minimal and minor. It is hoped that the style file will be converged to the final form rapidly. Nevertheless, there are specific issues which are on the "table" and they are described herein.

At this stage, several chapters are missing. The effects of the deviations from the ideal gas model on the properties should be included. Further topics related to non-ideal gas such as steam and various freons are in the process of being added to this book especially in relationship to Fanno flow.

One of the virtue of this book lay in the fact that it contains a software that is extensible. For example, the Fanno module can be extended to include effects of real gases. This part will be incorporated in the future hopefully with the help of others.

Specific missing parts from every chapters are discussed below. These omissions, mistakes, approach problems are sometime appears in the book under the Meta simple like this

Meta

sample this part.

Meta End

Questions/problems appear as a marginal note. On occasions a footnote was used to point out for a need of improvement. You are always welcome to add a new material: problem, question, illustration or photo of experiment. Material can be further illuminate. Additional material can be provided to give a different angle on the issue at hand.

Speed of Sound [beta]

- Discussion about the movement in medium with variation in speed of sound. This concept in relation of the wind tunnel
- Problems with atmosphere with varied density and temperature. Mixed gases and liquids. (some what done)
- More problems in relationship to two phase. Speed of sound in wet steam.

Stagnation effects [advance]

- Extend the applicability with examples.
- C_p as a function of temperature (deviation from ideal gas model) "real gas"' like water vapor
- History – on the teaching part (for example when the concept of stagnation was first taught).

Nozzle [advance]

- The effect of external forces (add problems).
- Real gases effects (mere temperature effects)
- Flow with "tabulated gases" calculations
- Phase change and two phase flow (multi choking points) effects (after 1.0 version).
- The dimensional analysis of the flow when the flow can be considered as isothermal.
- The combined effects of isentropic nozzle with heat transfer (especially with relationship to the program.).

Normal Shock [advance]

- Extend the partially (open/close) moving shock theory. [done]
- Provide more examples on the previous topic.
- Shock in real gases like water vapor.
- Shock in (partially) two phase gases like air with dust particles.
- Extend the shock tube [almost done]
- Shocks in two and three dimensions.

Minor Loss [NSV]

- Abrupt expansion
- Flow in Bend

Isothermal Flow [advance]

- Classification of Problems
- Comparison of results with Fanno flow
- Pipes Network calculations.

Fanno Flow [advance]

- More examples: various categories
- Some improvement on the software (clean up)
- Real gas effects (compressible factor)
- Tabulated gas

Rayleigh Flow [beta]

- To mature the chapter: discussion on the "dark" corners of this model.
- Provide discussion on variations of the effecting parameters.
- Examples: provide categorization

Add mass [NSY]

- Simple add mass in a continuous form

Evacuation and filling semi rigid Chambers [alpha]

- To construct the Rayleigh flow in the tube (thermal chocking)
- Energy equation (non isentropic process)
- Examples classifications
- Software (converting the FORTRAN program to c++)

Evacuating and filling chambers under external forces [alpha]

- Comparison with chemical reaction case
- Energy equation (non isentropic process)
- Examples
- Software transformation from FORTRAN to c++. The FORTRAN version will not be included.

Oblique Shock [advance]

- Add application to design problems
- Real Gas effects

Prandtl–Meyer

- The limitations (Prandtl-Meyer) (done).
- Application
- Cylindrical coordinate
- Marcell–Taylor (from the notes)
- Examples

Transient problem [NYP]

- Method of Characteristic

General 1-D flow [NYP]

CHAPTER 1

Introduction

1.1 What is Compressible Flow?

This book deals with an introduction[1] to the flow of compressible substances (gases). The main difference between compressible flow and almost incompressible flow is not the fact that compressibility has to be considered. Rather, the difference is in two phenomena that do not exist in incompressible flow[2]. The first phenomenon is the very sharp discontinuity (jump) in the flow in properties. The second phenomenon is the choking of the flow. Choking is when downstream variations don't effect the flow[3]. Though choking occurs in certain pipe flows in astronomy, there also are situations of choking in general (external) flow[4]. Choking is referred to as the situation where downstream conditions, which are beyond a critical value(s), doesn't affect the flow.

The shock wave and choking are not intuitive for most people. However, one has to realize that intuition is really a condition where one uses his past experiences to predict other situations. Here one has to learn to use his intuition as a tool for future use. Thus, not only aeronautic engineers, but other engineers, and even manufacturing engineers will be able use this "intuition" in design and even research.

[1] This book is gradually sliding to include more material that isn't so introductory. But an attempt is made to present the material in introductory level.

[2] It can be argued that in open channel flow there is a hydraulic jump (discontinuity) and in some ranges no effect of downstream conditions on the flow. However, the uniqueness of the phenomena in the gas dynamics provides spectacular situations of a limited length (see Fanno model) and thermal choking, etc. Further, there is no equivalent to oblique shock wave. Thus, this richness is unique to gas dynamics.

[3] The thermal choking is somewhat different but a similarity exists.

[4] This book is intended for engineers and therefore a discussion about astronomical conditions isn't presented.

1.2 Why Compressible Flow is Important?

Compressible flow appears in many natural and many technological processes. Compressible flow deals with more than air, including steam, natural gas, nitrogen and helium, etc. For instance, the flow of natural gas in a pipe system, a common method of heating in the u.s., should be considered a compressible flow. These processes include the flow of gas in the exhaust system of an internal combustion engine, and also gas turbine, a problem that led to the Fanno flow model. The above flows that were mentioned are called internal flows. Compressible flow also includes flow around bodies such as the wings of an airplane, and is considered an external flow.

These processes include situations not expected to have a compressible flow, such as manufacturing process such as the die casting, injection molding. The die casting process is a process in which liquid metal, mostly aluminum, is injected into a mold to obtain a near final shape. The air is displaced by the liquid metal in a very rapid manner, in a matter of milliseconds, therefore the compressibility has to be taken into account.

Clearly, Aero Engineers are not the only ones who have to deal with some aspect of compressible flow. For manufacturing engineers there are many situations where the compressibility or compressible flow understating is essential for adequate design. For instance, the control engineers who are using pneumatic systems use compressed substances. The cooling of some manufacturing systems and design of refrigeration systems also utilizes compressed air flow knowledge. Some aspects of these systems require consideration of the unique phenomena of compressible flow.

Traditionally, most gas dynamics (compressible flow) classes deal mostly with shock waves and external flow and briefly teach Fanno flows and Rayleigh flows (two kind of choking flows). There are very few courses that deal with isothermal flow. In fact, many books on compressible flow ignore the isothermal flow[5] .

In this book, a greater emphasis is on the internal flow. This doesn't in any way meant that the important topics such as shock wave and oblique shock wave should be neglected. This book contains several chapters which deal with external flow as well.

1.3 Historical Background

In writing this book it became clear that there is more unknown and unwritten about the history of compressible fluid than known. While there are excellent books about the history of fluid mechanics (hydraulic) see for example book by Rouse[6]. There are numerous sources dealing with the history of flight and airplanes (aeronautic)[7]. Aeronautics is an overlapping part of compressible flow, however these two fields are different. For example, the Fanno flow and isothermal flow, which are the core of

[5]Any search on the web on classes of compressible flow will show this fact and the undersigned can testify that this was true in his first class as a student of compressible flow.

[6]Hunter Rouse and Simon Inc, History of Hydraulics (Iowa City: Institute of Hydraulic Research, 1957)

[7]Anderson, J. D., Jr. 1997. A History of Aerodynamics: And Its Impact on Flying Machines, Cambridge University Press, Cambridge, England.

gas dynamics, are not part of aerodynamics. Possible reasons for the lack of written documentation are one, a large part of this knowledge is relatively new, and two, for many early contributors this topic was a side issue. In fact, only one contributor of the three main models of internal compressible flow (Isothermal, Fanno, Rayleigh) was described by any text book. This was Lord Rayleigh, for whom the Rayleigh flow was named. The other two models were, to the undersigned, unknown. Furthermore, this author did not find any reference to isothermal flow model earlier to Shapiro's book. There is no book[8] that describes the history of these models. For instance, the question, who was Fanno, and when did he live, could not be answered by any of the undersigned's colleagues in University of Minnesota or elsewhere.

At this stage there are more questions about the history of compressible flow needing to be answered. Sometimes, these questions will appear in a section with a title but without text or with only a little text. Sometimes, they will appear in a footnote like this[9] For example, it is obvious that Shapiro published the erroneous conclusion that all the chocking occurred at $M = 1$ in his article which contradicts his isothermal model. Additional example, who was the first to "conclude" the "all" the chocking occurs at $M = 1$? Is it Shapiro?

Originally, there was no idea that there are special effects and phenomena of compressible flow. Some researchers even have suggested that compressibility can be "swallowed" into the ideal flow (Euler's equation's flow is sometimes referred to as ideal flow). Even before Prandtl's idea of boundary layer appeared, the significant and importance of compressibility emerged.

In the first half of nineteen century there was little realization that the compressibility is important because there were very little applications (if any) that required the understanding of this phenomenon. As there were no motivations to investigate the shock wave or choked flow both were treated as the same, taking compressible flow as if it were incompressible flow.

It must be noted that researchers were interested in the speed of sound even long before applications and knowledge could demand any utilization. The research and interest in the speed of sound was a purely academic interest. The early application in which compressibility has a major effect was with fire arms. The technological improvements in fire arms led to a gun capable of shooting bullets at speeds approaching to the speed of sound. Thus, researchers were aware that the speed of sound is some kind of limit.

In the second half of the nineteen century, Mach and Fliegner "stumbled" over the shock wave and choking, respectively. Mach observed shock and Fliegner measured the choking but theoretical science did not provide explanation for it (or was award that there is an explanation for it.).

In the twentieth century the flight industry became the pushing force. Understandably, aerospace engineering played a significant role in the development of this

[8]The only remark found about Fanno flow that it was taken from the Fanno Master thesis by his adviser. Here is a challenge: find any book describing the history of the Fanno model.

[9]Who developed the isothermal model? The research so far leads to Shapiro. Perhaps this flow should be named after the Shapiro. Is there any earlier reference to this model?

knowledge. Giants like Prandtl and his students like Von Karman , as well as others like Shapiro , dominated the field. During that time, the modern basic classes became "solidified." Contributions by researchers and educators from other fields were not as dominant and significant, so almost all text books in this field are written from an aerodynamic prospective.

1.3.1 Early Developments

The compressible flow is a subset of fluid mechanics/hydraulics and therefore the knowledge development followed the understanding of incompressible flow. Early contributors were motivated from a purely intellectual curiosity, while most later contributions were driven by necessity. As a result, for a long time the question of the speed of sound was bounced around.

Speed of Sound

The idea that there is a speed of sound and that it can be measured is a major achievement. A possible explanation to this discovery lies in the fact that mother nature exhibits in every thunder storm the difference between the speed of light and the speed of sound. There is no clear evidence as to who came up with this concept, but some attribute it to Galileo Galilei: 166x. Galileo, an Italian scientist, was one of the earliest contributors to our understanding of sound. Dealing with the difference between the two speeds (light, sound) was a major part of Galileo's work. However, once there was a realization that sound can be measured, people found that sound travels in different speeds through different mediums. The early approach to the speed of sound was by the measuring of the speed of sound.

Other milestones in the speed of sound understanding development were by Leonardo Da Vinci, who discovered that sound travels in waves (1500). Marin Mersenne was the first to measure the speed of sound in air (1640). Robert Boyle discovered that sound waves must travel in a medium (1660) and this lead to the concept that sound is a pressure change. Newton was the first to formulate a relationship between the speed of sound in gases by relating the density and compressibility in a medium (by assuming isothermal process). Newton's equation is missing the heat ratio, k (late 1660's). Maxwell was the first to derive the speed of sound for gas as $c = \sqrt{k\,R\,T}$ from particles (statistical) mechanics. Therefore some referred to coefficient $\sqrt{k}$ as Maxwell's coefficient.

1.3.2 The shock wave puzzle

Here is where the politics of science was a major obstacle to achieving an advancement[10]. not giving the due credit to Rouse. In the early 18xx, conservation of energy was a concept that was applied only to mechanical energy. On the other side, a different group of scientists dealt with calorimetry (internal energy). It was easier to publish articles about the second law of thermodynamics than to convince anyone of the first law of thermodynamics. Neither of these groups would agree to "merge" or "relinquish" control of their "territory" to the other. It took about a century to establish the first law[11].

At first, Poisson found a "solution" to the Euler's equations with certain boundary conditions which required discontinuity[12] which had obtained an implicit form in 1808. Poisson showed that solutions could approach a discontinuity by using conservation of mass and momentum. He had then correctly derived the jump conditions that discontinuous solutions must satisfy. Later, Challis had noticed contradictions concerning some solutions of the equations of compressible gas dynamics[13]. Again the "jumping" conditions were redeveloped by two different researchers independently: Stokes and Riemann. Riemann, in his 1860 thesis, was not sure whether or not discontinuity is only a mathematical creature or a real creature. Stokes in 1848 retreated from his work and wrote an apology on his "mistake."[14] Stokes was convinced by Lord Rayleigh and Lord Kelvin that he was mistaken on the grounds that energy is conserved (not realizing the concept of internal energy).

At this stage some experimental evidence was needed. Ernest Mach studied several fields in physics and also studied philosophy. He was mostly interested in experimental physics. The major breakthrough in the understanding of compressible flow came when Ernest Mach "stumbled" over the discontinuity. It is widely believed that Mach had done his research as purely intellectual research. His research centered on optic aspects which lead him to study acoustic and therefore supersonic flow (high speed, since no Mach number was known at that time). However, it is logical to believe that his interest had risen due to the need to achieve powerful/long–distance shooting

[10]Amazingly, science is full of many stories of conflicts and disputes. Aside from the conflicts of scientists with the Catholic Church and Muslim religion, perhaps the most famous is that of Newton's netscaping (stealing and embracing) Leibniz['s] invention of calculus. There are even conflicts from not giving enough credit, like Moody. Even the undersigned encountered individuals who have tried to ride on his work. The other kind of problem is "hijacking" by a sector. Even on this subject, the Aeronautic sector "took over" gas dynamics as did the emphasis on mathematics like perturbations methods or asymptotic expansions instead on the physical phenomena. Major material like Fanno flow isn't taught in many classes, while many of the mathematical techniques are currently practiced. So, these problems are more common than one might be expected.

[11]This recognition of the first law is today the most "obvious" for engineering students. Yet for many it was still debatable up to the middle of the nineteen century.

[12]Siméon Denis Poisson, French mathematician, 1781-1840 worked in Paris, France. "M'emoire sur la th'eorie du son," J. Ec. Polytech. 14 (1808), 319-392. From Classic Papers in Shock Compression Science, 3-65, High-press. Shock Compression Condens. Matter, Springer, New York, 1998.

[13]James Challis, English Astronomer, 1803-1882. worked at Cambridge, England UK. "On the velocity of sound," Philos. Mag. XXXII (1848), 494-499

[14]Stokes George Gabriel Sir, Mathematical and Physical Papers, Reprinted from the original journals and transactions, with additional notes by the author. Cambridge, University Press, 1880-1905.

rifles/guns. At that time many inventions dealt with machine guns which were able to shoot more bullets per minute. At the time, one anecdotal story suggests a way to make money by inventing a better killing machine for the Europeans. While the machine gun turned out to be a good killing machine, defense techniques started to appear such as sand bags. A need for bullets that could travel faster to overcome these obstacles was created. Therefore, Mach's paper from 1876 deals with the flow around bullets. Nevertheless, no known[15] equations or explanations resulted from these experiments.

Mach used his knowledge in Optics to study the flow around bullets. What makes Mach's achievement all the more remarkable was the technique he used to take the historic photograph: He employed an innovative approach called the shadowgraph. He was the first to photograph the shock wave. In his paper discussing "Photographische Fixierung der durch Projektile in der Luft eingeleiten Vorgange" he showed a picture of a shock wave (see Figure 1.7). He utilized the variations of the air density to clearly show shock line at the front of the bullet. Mach had good understanding of the fundamentals of supersonic flow and the effects on bullet movement (supersonic flow). Mach's paper from 1876 demonstrated shock wave (discontinuity) and suggested the importance of the ratio of the velocity to the speed of sound. He also observed the existence of a conical shock wave (oblique shock wave).

Mach's contributions can be summarized as providing an experimental proof to discontinuity. He further showed that the discontinuity occurs at $M = 1$ and realized that the velocity ratio (Mach number), and not the velocity, is the important parameter in the study of the compressible flow. Thus, he brought confidence to the theoreticians to publish their studies. While Mach proved shock wave and oblique shock wave existence, he was not able to analyze it (neither was he aware of Poisson's work or the works of others.).

Back to the pencil and paper, the jump conditions were redeveloped and now named after Rankine[16] and Hugoniot[17]. Rankine and Hugoniot, redeveloped independently the equation that governs the relationship of the shock wave. Shock was assumed to be one dimensional and mass, momentum, and energy equations[18] lead to a solution which ties the upstream and downstream properties. What they could not prove or find was that shock occurs only when upstream is supersonic, i.e., direction of the flow. Later, others expanded Rankine-Hugoniot's conditions to a more general form[19].

[15] The words "no known" refer to the undersigned. It is possible that some insight was developed but none of the documents that were reviewed revealed it to the undersigned.

[16] William John Macquorn Rankine, Scottish engineer, 1820-1872. He worked in Glasgow, Scotland UK. "On the thermodynamic theory of waves of finite longitudinal disturbance," Philos. Trans. 160 (1870), part II, 277-288. Classic papers in shock compression science, 133-147, High-press. Shock Compression Condens. Matter, Springer, New York, 1998

[17] Pierre Henri Hugoniot, French engineer, 1851-1887. "Sur la propagation du mouvement dans les corps et sp'ecialement dans les gaz parfaits, I, II" J. Ec. Polytech. 57 (1887), 3-97, 58 (1889), 1-125. Classic papers in shock compression science, 161-243, 245-358, High-press. Shock Compression Condens. Matter, Springer, New York, 1998

[18] Today it is well established that shock has three dimensions but small sections can be treated as one dimensional.

[19] To add discussion about the general relationships.

Here, the second law has been around for over 40 years and yet the significance of it was not was well established. Thus, it took over 50 years for Prandtl to arrive at and to demonstrate that the shock has only one direction[20]. Today this equation/condition is known as Prandtl's equation or condition (1908). In fact Prandtl is the one who introduced the name of Rankine-Hugoniot's conditions not aware of the earlier developments of this condition. Theodor Meyer (Prandtl's student) derived the conditions for oblique shock in 1908[21] as a byproduct of the expansion work.

It was probably later that Stodola (Fanno's adviser) realized that the shock is the intersection of the Fanno line with the Rayleigh line. Yet, the supersonic branch is missing from his understanding (see Figure (1.1)). In fact, Stodola suggested the graphical solution utilizing the Fanno line.

The fact that the conditions and direction were known did not bring the solution to the equations. The "last nail" of understanding was put by Landau, a Jewish scientist who worked in Moscow University in the 1960's during the Communist regimes. A solution was found by Landau & Lifshitz and expanded by Kolosnitsyn & Stanyukovich (1984).

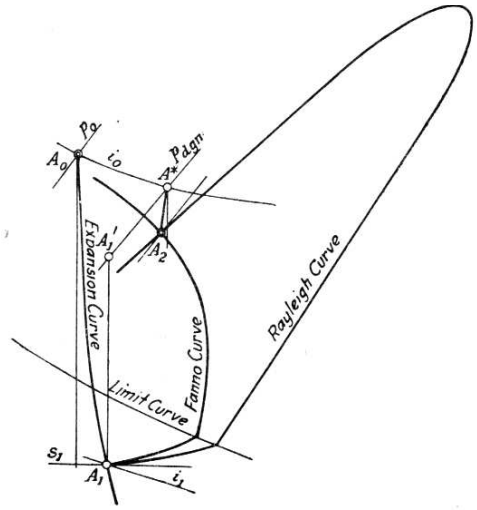

FIG. 51.—"Condition loci" for compression shock

Fig. -1.1. The shock as a connection of Fanno and Rayleigh lines after Stodola, Steam and Gas Turbine

Since early in the 1950s the analytical relationships between the oblique shock, deflection angle, shock angle, and Mach number was described as impossible to obtain. There were until recently (version 0.3 of this book) several equations that tied various properties/quantities for example, the relationship between upstream Mach number and the angles. The first full analytical solution connecting the angles with upstream Mach number was published in this book version 0.3. The probable reason that analytical solution was not discovered earlier because the claim in the famous report of NACA

[20] Some view the work of G. I. Taylor from England as the proof (of course utilizing the second law)
[21] Theodor Meyer in Mitteil. üb. Forsch-Arb. Berlin, 1908, No. 62, page 62.

1135 that explicit analytical solution isn't possible[22][23].

The question whether the angle of the oblique shock is stable or which of the three roots is stable was daunting since the early discovery that there are three possible solutions. It is amazing that early research concluded that only the weak solution is possible or stable as opposed to the reality. The first that attempt this question where in 1931 by Epstein[24]. His analysis was based on Hamilton's principle when he ignore the boundary condition. The results of that analysis was that strong shock is unstable. The researchers understood that flow after a strong shock was governed by elliptic equation while the flow after a weak shock was governed by hyperbolic equations. This difference probably results in not recognizing that The boundary conditions play an important role in the stability of the shock[25]. In fact analysis based on Hamilton's principle isn't suitable for stability because entropy creation was recognized 1955 by Herivel[26].

Carrier[27] was first to recognize that strong and weak shocks stable. In fact, the confusion on this issue is persistent until now. Even all books that were published recently claimed that no strong shock was ever observed in flow around cone (Taylor–Maccoll flow). In fact, even this author sinned in this erroneous conclusion. The real question isn't if they exist rather under what conditions these shocks exist which was suggested by Courant and Friedrichs in their book "Supersonic Flow and Shock Waves," published by Interscience Publishers, Inc. New York, 1948, p. 317.

The effect of real gases was investigated very early since steam was used propel turbines. In general the mathematical treatment was left to numerical investigation and there is relatively very little known on the difference between ideal gas model and real gas. For example, recently, Henderson and Menikoff[28] dealt with only the procedure to find the maximum of oblique shock, but no comparison between real gases and ideal gas is offered there.

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[22]Since writing this book, several individuals point out that a solution was found in book "Analytical Fluid Dynamics" by Emanuel, George, second edition, December 2000 (US$ 124.90). That solution is based on a transformation of $\sin\theta$ to $\tan\beta$. It is interesting that transformation result in one of root being negative. While the actual solution all the roots are real and positive for the attached shock. The presentation was missing the condition for the detachment or point where the model collapse. But more surprisingly, similar analysis was published by Briggs, J. "Comment on Calculation of Oblique shock waves," AIAA Journal Vol 2, No 5 p. 974, 1963. Hence, Emanuel's partial solution just redone 36 years work (how many times works have to be redone in this field). In addition there was additional publishing of similar works by Mascitti, V.R. and Wolf, T. In a way, part of analysis of this book is also redoing old work. Yet, what is new in this work is completeness of all the three roots and the analytical condition for detached shock and breaking of the model.

[23]See for a longer story in www.potto.org/obliqueArticle.php.

[24]Epstein, P. S., "On the air resistance of Projectiles," Proceedings of the National Academy of Science, Vol. 17, 1931, pp. 532-547.

[25]In study this issue this author realized only after examining a colleague experimental Picture 14.4 that it was clear that the Normal shock along with strong shock and weak shock "live" together peacefully and in stable conditions.

[26]Herivel, J. F., "The Derivation of The Equations of Motion On an Ideal Fluid by Hamilton's Principle,," Proceedings of the Cambridge philosophical society, Vol. 51, Pt. 2, 1955, pp. 344-349.

[27]Carrier, G.F., "On the Stability of the supersonic Flows Past as a Wedge," Quarterly of Applied Mathematics, Vol. 6, 1949, pp. 367–378.

[28] Henderson and Menikoff, "Triple Shock Entropy Theorem," Journal of Fluid Mechanics 366 (1998) pp. 179–210.

The moving shock and shock tube were study even before World War Two (II). The realization that in most cases the moving shock can be analyzed as steady state since it approaches semi steady state can be traced early of 1940's. Up to this version 0.4.3 of this book (as far it is known, this book is the first to publish this tables), trial and error method was the only method to solve this problem. Only after the dimensionless presentation of the problem and the construction of the moving shock table the problem became trivial. Later, an explicit analytical solution for shock a head of piston movement (a special case of open valve) was originally published in this book for the first time.

1.3.3 Choking Flow

The choking problem is almost unique to gas dynamics and has many different forms. Choking wasn't clearly to be observed, even when researcher stumbled over it. No one was looking for or expecting the choking to occur, and when it was found the significance of the choking phenomenon was not clear. The first experimental choking phenomenon was discovered by Fliegner's experiments which were conducted some time in the middle of 186x[29] on air flow through a converging nozzle. As a result deLavel's nozzle was invented by Carl Gustaf Patrik de Laval in 1882 and first successful operation by another inventor (Curtis) 1896 used in steam turbine. Yet, there was no realization that the flow is choked just that the flow moves faster than speed of sound.

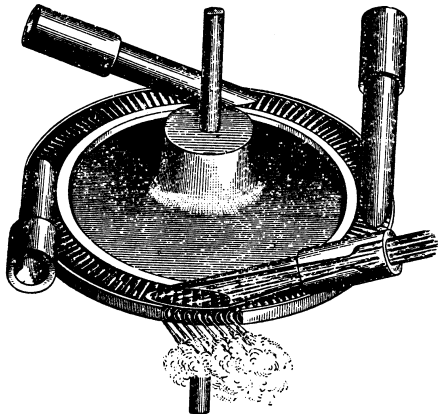

Fig. -1.2. The schematic of deLavel's turbine after Stodola, Steam and Gas Turbine

The introduction of the steam engine and other thermodynamics cycles led to the choking problem. The problem was introduced because people wanted to increase the output of the Engine by increasing the flames (larger heat transfer or larger energy) which failed, leading to the study and development of Rayleigh flow. According the thermodynamics theory (various cycles) the larger heat supply for a given temperature difference (larger higher temperature) the larger the output, but after a certain point it did matter (because the steam was choked). The first to discover (try to explain) the choking phenomenon was Rayleigh[30].

After the introduction of the deLavel's converging–diverging nozzle theoretical

[29] Fliegner Schweizer Bauztg., Vol 31 1898, p. 68–72. The theoretical first work on this issue was done by Zeuner, "Theorie die Turbinen," Leipzig 1899, page 268 f.

[30] Rayleigh was the first to develop the model that bears his name. It is likely that others had noticed that flow is choked, but did not produce any model or conduct successful experimental work.

work was started by Zeuner[31]. Later continue by Prandtl's group[32] starting 1904. In 1908 Meyer has extend this work to make two dimensional calculations[33]. Experimental work by Parenty[34] and others measured the pressure along the converging-diverging nozzle.

It was commonly believed[35] that the choking occurs only at $M = 1$. The first one to analyzed that choking occurs at $1/\sqrt{k}$ for isothermal flow was Shapiro (195x). It is so strange that a giant like Shapiro did not realize his model on isothermal contradict his conclusion from his own famous paper. Later Romer at el extended it to isothermal variable area flow (1955). In this book, this author adapts E.R.G. Ecert's idea of dimensionless parameters control which determines where the reality lay between the two extremes. Recently this concept was proposed (not explicitly) by Dutton and Converdill (1997)[36]. Namely, in many cases the reality is somewhere between the adiabatic and the isothermal flow. The actual results will be determined by the modified Eckert number to which model they are closer.

Nozzle Flow

The first "wind tunnel" was not a tunnel but a rotating arm attached at the center. At the end of the arm was the object that was under observation and study. The arm's circular motion could reach a velocity above the speed of sound at its end. Yet, in 1904 the Wright brothers demonstrated that results from the wind tunnel and spinning arm are different due to the circular motion. As a result, the spinning arm was no longer used in testing. Between the turn of the century and 1947-48, when the first supersonic wind tunnel was built, several models that explained choking at the throat have been built.

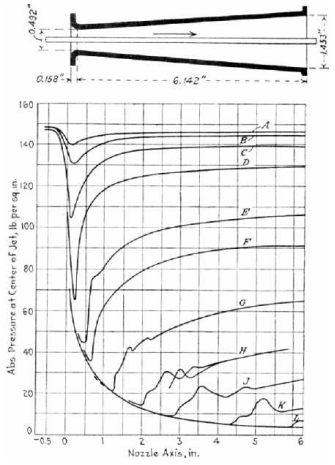

Fig. -1.3. The measured pressure in a nozzle taken from Stodola 1927 Steam and Gas Turbines

A different reason to study the converging-diverging nozzle was the Venturi meter

[31] Zeuner, "Theorie der Turbinen, Leipzig 1899 page 268 f.

[32] Some of the publications were not named after Prandtl but rather by his students like Meyer, Theodor. In the literature appeared reference to article by Lorenz in the Physik Zeitshr., as if in 1904. Perhaps, there are also other works that this author did not come a crossed.

[33] Meyer, Th., Über zweidimensionals Bewegungsvordange eines Gases, Dissertation 1907, erschienen in den Mitteilungen über Forsch.-Arb. Ing.-Wes. heft 62, Berlin 1908.

[34] Parenty, Comptes R. Paris, Vol. 113, 116, 119; Ann. Chim. Phys. Vol. 8. 8 1896, Vol 12, 1897.

[35] The personal experience of this undersigned shows that even instructors of Gas Dynamics are not aware that the chocking occurs at different Mach number and depends on the model.

[36] These researchers demonstrate results between two extremes and actually proposed this idea. However, that the presentation here suggests that topic should be presented case between two extremes.

which was used in measuring the flow rate of gases. Bendemann [37] carried experiments to study the accuracy of these flow meters and he measured and refound that the flow reaches a critical value (pressure ratio of 0.545) that creates the maximum flow rate.

There are two main models or extremes that describe the flow in the nozzle: isothermal and adiabatic.

Romer et al[38] analyzed the isothermal flow in a nozzle. It is remarkable that choking was found as $1/\sqrt{k}$ as opposed to one (1). In general when the model is assumed to be isothermal the choking occurs at $1/\sqrt{k}$. The concept that the choking point can move from the throat introduced by[39] a researcher unknown to this author. It is very interesting that the isothermal nozzle was proposed by Romer at el 1955 (who was behind the adviser or the student?). These researchers were the first ones to realized that choking can occurs at different Mach number $(1/\sqrt{k})$ other than the isothermal pipe.

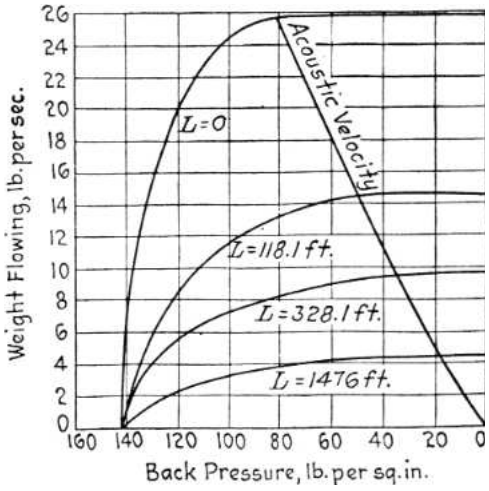

Fig. -1.4. Flow rate as a function of the back pressure taken after Stodola 1927 Steam and Gas Turbines

Rayleigh Flow

Rayleigh was probably[40], the first to suggest a model for frictionless flow with a constant heat transfer. Rayleigh's work was during the time when it was debatable as to whether there are two forms of energies (mechanical, thermal), even though Watt and others found and proved that they are the same. Therefore, Rayleigh looked at flow without mechanical energy transfer (friction) but only thermal energy transfer. In Rayleigh flow, the material reaches choking point due to heat transfer, hence the term "thermally choked" is used; no additional heat to "flow" can occur.

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[37] Bendemann Mitteil über Forschungsarbeiten, Berlin, 1907, No. 37.

[39] Romer, I Carl Jr., and Ali Bulent Cambel, "Analysis of Isothermal Variable Area Flow," Aircraft Eng. vol. 27 no 322, p. 398 December 1955.

[39] This undersign didn't find the actual trace to the source of proposing this effect. However, some astronomy books showing this effect in a dimensional form without mentioning the original researcher. In dimensionless form, this phenomenon produces a dimensionless number similar to Ozer number and therefor the name Ozer number adapted in this book.

[40] As most of the history research has shown, there is also a possibility that someone found it earlier. For example, Simeon–Denis Poisson was the first one to realize the shock wave possibility.

Fanno Flow

The most important model in compressible flow was suggested by Gino Fanno in his Master's thesis (1904). The model bears his name. Yet, according to Dr. Rudolf Mumenthaler from UTH University (the place where Fanno Studied), no copy of the thesis can be found in the original University and perhaps only in the personal custody of the Fanno family[41]. Fanno attributes the main pressure reduction to friction. Thus, flow that is dominantly adiabatic could be simplified and analyzed. The friction factor is the main component in the analysis as Darcy f[42] had already proposed in 1845. The arrival of the Moody diagram, which built on Hunter Rouse's (194x) work made Darcy–Weisbach's equation universally useful. Without the existence of the friction factor data, the Fanno model wasn't able to produce a prediction useful for the industry. Additionally an understating of the supersonic branch of the flow was unknown (The idea of shock in tube was not raised at that time.). Shapiro organized all the material in a coherent way and made this model useful.

Meta

Did Fanno realize that the flow is choked? It appears at least in Stodola's book that choking was understood in 1927 and even earlier. The choking was assumed only to be in the subsonic flow. But because the actual Fanno's thesis is not available, the question cannot be answered yet. When was Gas Dynamics (compressible flow) as a separate class started? Did the explanation for the combination of diverging-converging nuzzle with tube for Fanno flow first appeared in Shapiro's book?

Meta End

Isothermal Flow

The earliest reference to isothermal flow was found in Shapiro's Book. The model suggests that the choking occurs at $1/\sqrt{k}$ and it appears that Shapiro was the first one to realize this difference compared to the other models. In reality, the flow is choked somewhere between $1/\sqrt{k}$ to one for cases that are between Fanno (adiabatic) and isothermal flow. This fact was evident in industrial applications where the expectation of the choking is at Mach one, but can be explained by choking at a lower Mach number. No experimental evidence, known by the undersigned, was ever produced to verify this finding.

1.3.4 External flow

When the flow over an external body is about .8 Mach or more the flow must be considered to be a compressible flow. However at a Mach number above 0.8 (relative of velocity of the body to upstream velocity) a local Mach number (local velocity) can

[41] This material is very important and someone should find it and make it available to researchers.
[42] Fanning f based radius is only one quarter of the Darcy f which is based on diameter

reach $M = 1$. At that stage, a shock wave occurs which increases the resistance. The Navier-Stokes equations which describe the flow (or even Euler equations) were considered unsolvable during the mid 18xx because of the high complexity. This problem led to two consequences. Theoreticians tried to simplify the equations and arrive at approximate solutions representing specific cases. Examples of such work are Hermann von Helmholtz's concept of vortex filaments (1858), Lanchester's concept of circulatory flow (1894), and the Kutta-Joukowski circulation theory of lift (1906). Practitioners like the Wright brothers relied upon experimentation to figure out what theory could not yet tell them.

Ludwig Prandtl in 1904 explained the two most important causes of drag by introducing the boundary layer theory. Prandtl's boundary layer theory allowed various simplifications of the Navier-Stokes equations. Prandtl worked on calculating the effect of induced drag on lift. He introduced the *lifting line theory*, which was published in 1918-1919 and enabled accurate calculations of induced drag and its effect on lift[43].

During World War I, Prandtl created his thin–airfoil theory that enabled the calculation of lift for thin, cambered airfoils. He later contributed to the Prandtl-Glauert rule for subsonic airflow that describes the compressibility effects of air at high speeds. Prandtl's student, Von Karman reduced the equations for supersonic flow into a single equation.

After the First World War aviation became important and in the 1920s a push of research focused on what was called the *compressibility problem*. Airplanes could not yet fly fast, but the propellers (which are also airfoils) did exceed the speed of sound, especially at the propeller tips, thus exhibiting inefficiency. Frank Caldwell and Elisha Fales demonstrated in 1918 that at a critical speed (later renamed the *critical Mach number*) airfoils suffered dramatic increases in drag and decreases in lift. Later, Briggs and Dryden showed that the problem was related to the shock wave. Meanwhile in Germany, one of Prandtl's assistants, J. Ackeret, simplified the shock equations so that they became easy to use. After World War Two, the research had continued and some technical solutions were found. Some of the solutions lead to tedious calculations which lead to the creation of Computational Fluid Dynamics (CFD). Today these methods of perturbations and asymptotic are hardly used in wing calculations[44]. That is the "dinosaur[45]" reason that even today some instructors are teaching mostly the perturbations and asymptotic methods in Gas Dynamics classes.

More information on external flow can be found in , John D. Anderson's Book "History of Aerodynamics and Its Impact on Flying Machines," Cambridge University Press, 1997.

[43] The English call this theory the Lanchester–Prandtl theory. This is because the English Astronomer Frederick Lanchester published the foundation for Prandtl's theory in his 1907 book *Aerodynamics*. However, Prandtl claimed that he was not aware of Lanchester's model when he had begun his work in 1911. This claim seems reasonable in the light that Prandtl was not ware of earlier works when he named erroneously the conditions for the shock wave. See for the full story in the shock section.

[44] This undersigned is aware of only one case that these methods were really used to calculations of wing.

[45] It is like teaching using slide ruler in today school. By the way, slide rule is sold for about 7.5$ on the net. Yet, there is no reason to teach it in a regular school.

1.3.5 Filling and Evacuating Gaseous Chambers

It is remarkable that there were so few contributions made in the area of a filling or evacuation gaseous chamber. The earlier work dealing with this issue was by Giffen, 1940, and was republished by Owczarek, J. A., the model and solution to the nozzle attached to chamber issue in his book "Fundamentals of Gas Dynamics."[46]. He also extended the model to include the unchoked case. Later several researchers mostly from the University in Illinois extended this work to isothermal nozzle (choked and unchoked).

The simplest model of nozzle, is not sufficient in many cases and a connection by a tube (rather just nozzle or orifice) is more appropriated. Since World War II considerable works have been carried out in this area but with very little progress[47]. In 1993 the first reasonable models for forced volume were published by the undersigned. Later, that model was extended by several research groups, The analytical solution for forced volume and the "*balloon*" problem (airbag's problem) model were published first in this book (version 0.35) in 2005. The classification of filling or evacuating the chamber as external control and internal control (mostly by pressure) was described in version 0.3 of this book by this author.

1.3.6 Biographies of Major Figures

In this section a short summary of major figures that influenced the field of gas dynamics is present. There are many figures that should be included and a biased selection was required. Much information can be obtained from other resources, such as the Internet. In this section there is no originality and none should be expected.

Fig. -1.5. Portrait of Galileo Galilei

Galileo Galilei

Galileo was born in Pisa, Italy on February 15, 1564 to musician Vincenzo Galilei and Giulia degli Ammannati. The oldest of six children, Galileo moved with his family in early 1570 to Florence. Galileo started his studying at the University of Pisa in 1581. He then became a professor of mathematics at the University of Padua in 1592. During the time after his study, he made numerous discoveries such as that of the pendulum clock, (1602). Galileo also proved that objects fell with the same velocity regardless of their size.

[46] International Textbook Co., Scranton, Pennsylvania, 1964.

[47] In fact, the emergence of the CFD gave the illusion that there are solutions at hand, not realizing that garbage in is garbage out, i.e., the model has to be based on scientific principles and not detached from reality. As anecdotal story explaining the lack of progress, in die casting conference there was a discussion and presentation on which turbulence model is suitable for a *complete* still liquid. Other "strange" models can be found in the undersigned's book "Fundamentals of Die Casting Design.

Galileo had a relationship with Marina Gamba (they never married)ho lived and worked in his house in Padua, where she bore him three children. However, this relationship did not last and Marina married Giovanni Bartoluzzi and Galileo's son, Vincenzio, joined him in Florence (1613).

Galileo invented many mechanical devices such as the pump and the telescope (1609). His telescopes helped him make many astronomic observations which proved the Copernican system. Galileo's observations got him into trouble with the Catholic Church, however, because of his noble ancestry, the church was not harsh with him. Galileo was convicted after publishing his book *Dialogue*, and he was put under house arrest for the remainder of his life. Galileo died in 1642 in his home outside of Florence.

Ernest Mach (1838-1916)

Ernest Mach was born in 1838 in Chrlice (now part of Brno), when Czechia was still a part of the Austro–Hungary empire. Johann, Mach's father, was a high school teacher who taught Ernest at home until he was 14, when he studied in Kromeriz Gymnasium, before he entered the university of Vienna were he studies mathematics, physics and philosophy. He graduated from Vienna in 1860. There Mach wrote his thesis "On Electrical Discharge and Induction." Mach was interested also in physiology of sensory perception.

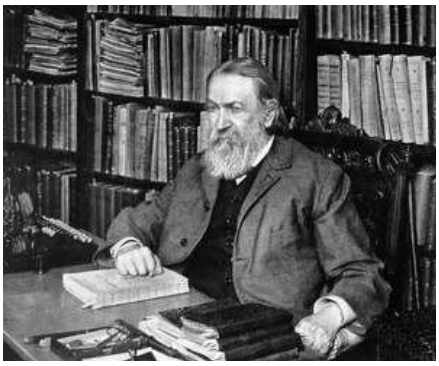


Fig. -1.6. Photo of Ernest Mach

At first he received a professorship position at Graz in mathematics (1864) and was then offered a position as a professor of surgery at the university of Salzburg, but he declined. He then turned to physics, and in 1867 he received a position in the Technical University in Prague[48] where he taught experimental physics for the next 28 years.

Mach was also a great thinker/philosopher and influenced the theory of relativity dealing with frame of reference. In 1863, Ernest Mach (1836 - 1916) published Die Machanik in which he formalized this argument. Later, Einstein was greatly influenced by it, and in 1918, he named it *Mach's Principle*. This was one of the primary sources of inspiration for Einstein's theory of General Relativity.

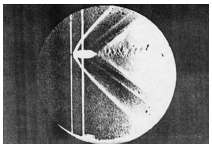


Fig. -1.7. The photo of the bullet in a supersonic flow which was taken by Mach. Note it was not taken in a wind tunnel

Mach's revolutionary experiment demon-

[48] It is interesting to point out that Prague provided us two of the top influential researchers: E. Mach and E.R.G. Eckert.

strated the existence of the shock wave as shown in Figure 1.7. It is amazing that Mach was able to photograph the phenomenon using the spinning arm technique (no wind tunnel was available at that time and most definitely nothing that could take a photo at supersonic speeds. His experiments required exact timing. He was not able to attach the camera to the arm and utilize the remote control (not existent at that time). Mach's shadowgraph technique and a related method called *Schlieren Photography* are still used today.

Yet, Mach's contributions to supersonic flow were not limited to experimental methods alone. Mach understood the basic characteristics of external supersonic flow where the most important variable affecting the flow is the ratio of the speed of the flow[49] (U) relative to the speed of sound (c). Mach was the first to note the transition that occurs when the ratio U/c goes from being less than 1 to greater than 1. The name Mach Number (M) was coined by J. Ackeret (Prandtl's student) in 1932 in honor of Mach.

John William Strutt (Lord Rayleigh)

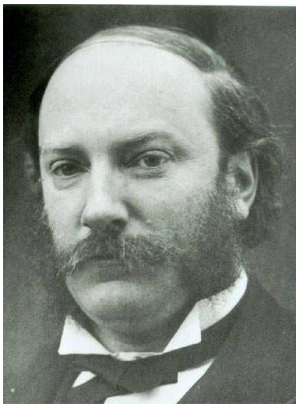


Fig. -1.8. Lord Rayleigh portrait.

A researcher with a wide interest, started studies in compressible flow mostly from a mathematical approach. At that time there wasn't the realization that the flow could be choked. It seems that Rayleigh was the first who realized that flow with chemical reactions (heat transfer) can be choked.

Lord Rayleigh was a British physicist born near Maldon, Essex, on November 12, 1842. In 1861 he entered Trinity College at Cambridge, where he commenced reading mathematics. His exceptional abilities soon enabled him to overtake his colleagues. He graduated in the Mathematical Tripos in 1865 as Senior Wrangler and Smith's Prizeman. In 1866 he obtained a fellowship at Trinity which he held until 1871, the year of his marriage. He served for six years as the president of the government committee on explosives, and from 1896 to 1919 he acted as Scientific Adviser to Trinity House. He was Lord Lieutenant of Essex from 1892 to 1901.

Lord Rayleigh's first research was mainly mathematical, concerning optics and vibrating systems, but his later work ranged over almost the whole field of physics, covering sound, wave theory, color vision, electrodynamics, electromagnetism, light scattering, flow of liquids, hydrodynamics, density of gases, viscosity, capillarity, elasticity, and photography. Rayleigh's later work was concentrated on electric and magnetic problems. Rayleigh was considered to be an excellent instructor. His Theory of Sound was published in two volumes during 1877-1878, and his other extensive studies are re-

[49] Mach dealt with only air, but it is reasonable to assume that he understood that this ratio was applied to other gases.

ported in his scientific papers, six volumes issued during 1889-1920. Rayleigh was also a contributor to the Encyclopedia Britannica. He published 446 papers which, reprinted in his collected works, clearly show his capacity for understanding everything just a little more deeply than anyone else. He intervened in debates of the House of Lords only on rare occasions, never allowing politics to interfere with science. Lord Rayleigh, a Chancellor of Cambridge University, was a Justice of the Peace and the recipient of honorary science and law degrees. He was a Fellow of the Royal Society (1873) and served as Secretary from 1885 to 1896, and as President from 1905 to 1908. He received the Nobel Prize in 1904.

In 1871 he married Evelyn, sister of the future prime minister, the Earl of Balfour (of the famous Balfour declaration of the Jewish state). They had three sons, the eldest of whom was to become a professor of physics at the Imperial College of Science and Technology, London.

As a successor to James Clerk Maxwell, he was head of the Cavendish Laboratory at Cambridge from 1879-1884, and in 1887 became Professor of Natural Philosophy at the Royal Institute of Great Britain. Rayleigh died on June 30, 1919 at Witham, Essex.

William John Macquorn Rankine

William John Macquorn Rankine (July 2, 1820 - December 24, 1872) was a Scottish engineer and physicist. He was a founding contributor to the science of thermodynamics (Rankine Cycle). Rankine developed a theory of the steam engine. His steam engine manuals were used for many decades.

Fig. -1.9. Portrait of Rankine.

Rankine was well rounded interested beside the energy field he was also interested in civil engineering, strength of materials, and naval engineering in which he was involved in applying scientific principles to building ships.

Rankine was born in Edinburgh to British Army lieutenant David Rankine and Barbara Grahame, Rankine. As Prandtl due health reasons (only his own) Rankine initially had home schooling only later attended public eduction for a short period of time such as High School of Glasgow (1830). Later his family to Edinburgh and in 1834 he studied at a Military and Naval Academy. Rankine help his father who in the management and engineering of the Edinburgh and Dalkeith Railway. He never graduated from the University of Edinburgh (1838) and continue to work for Irish railroad for which he developed a technique, later known as Rankine's method, for laying out railway curves. In 1849 he found the relationship between saturated vapor pressure and temperature. Later he established relationships between the temperature, pressure and density of gases, and expressions for the latent heat of evaporation of a liquid.

Rankine never married, and his only brother and parents died before him. The history of Prandtl and Rankine suggest that home school (by a scientist) is much better than the public school.

Gino Girolamo Fanno

Fig. -1.10. The photo of Gino Fanno approximately in 1950.

Fanno a Jewish Engineer was born on November 18, 1888. He studied in a technical institute in Venice and graduated with very high grades as a mechanical engineer. Fanno was not as lucky as his brother, who was able to get into academia. Faced with anti–semitism, Fanno left Italy for Zurich, Switzerland in 1900 to attend graduate school for his master's degree. In this new place he was able to pose as a Roman Catholic, even though for short time he went to live in a Jewish home, Isaak Baruch Weil's family. As were many Jews at that time, Fanno was fluent in several languages including Italian, English, German, and French. He likely had a good knowledge of Yiddish and possibly some Hebrew. Consequently, he did not have a problem studying in a different language. In July 1904 he received his diploma (master). When one of Professor Stodola's assistants attended military service this temporary position was offered to Fanno. "Why didn't a talented guy like Fanno keep or obtain a position in academia after he published his model?" The answer is tied to the fact that somehow rumors about his roots began to surface. Additionally, the fact that his model was not a "smashing[50] success" did not help.

Later Fanno had to go back to Italy to find a job in industry. Fanno turned out to be a good engineer and he later obtained a management position. He married, and like his brother, Marco, was childless. He obtained a Ph.D. from Regian Istituto Superiore d'Ingegneria di Genova. However, on February 1939 Fanno was degraded (denounced) and he lost his Ph.D. (is this the first case in history) because of his Jewish nationality[51]. During the War (WWII), he had to be under house arrest to avoid being sent to the "vacation camps." To further camouflage himself, Fanno converted to Catholicism. Apparently, Fanno had a cache of old Italian currency (which was apparently still highly acceptable) which helped him and his wife survive the war. After the war, Fanno was only able to work in agriculture and agricultural engineering. Fanno passed way in 1960 without world recognition for his model.

Fanno's older brother, mentioned earlier Marco Fanno is a famous economist who later developed fundamentals of the supply and demand theory.

[50] Missing data about friction factor

[51] In some places, the ridicules claims that Jews persecuted only because their religion. Clearly, Fanno was not part of the Jewish religion (see his picture) only his nationality was Jewish.

Ludwig Prandtl

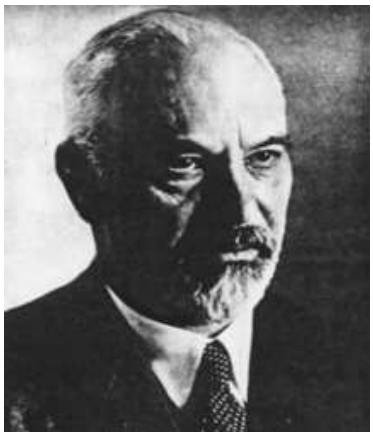


Fig. -1.11, Photo of Prandtl.

Perhaps Prandtl's greatest achievement was his ability to produce so many great scientists. It is mind boggling to look at the long list of those who were his students and colleagues. There is no one who educated as many great scientists as Prandtl. Prandtl changed the field of fluid mechanics and is called the modern father of fluid mechanics because of his introduction of boundary layer, turbulence mixing theories etc.

Ludwig Prandtl was born in Freising, Bavaria, in 1874. His father was a professor of engineering and his mother suffered from a lengthy illness. As a result, the young Ludwig spent more time with his father which made him interested in his father's physics and machinery books. This upbringing fostered the young Prandtl's interest in science and experimentation.

Prandtl started his studies at the age of 20 in Munich, Germany and he graduated at the age of 26 with a Ph.D. Interestingly, his Ph.D. was focused on solid mechanics. His interest changed when, in his first job, he was required to design factory equipment that involved problems related to the field of fluid mechanics (a suction device). Later he sought and found a job as a professor of mechanics at a technical school in Hannover, Germany (1901). During this time Prandtl developed his boundary layer theory and studied supersonic fluid flows through nozzles. In 1904, he presented the revolutionary paper "Flussigkeitsbewegung Bei Sehr Kleiner Reibung" (Fluid Flow in Very Little Friction), the paper which describes his boundary layer theory.

His 1904 paper raised Prandtl's prestige. He became the director of the Institute for Technical Physics at the University of Göttingen. He developed the Prandtl-Glauert rule for subsonic airflow. Prandtl, with his student Theodor Meyer, developed the first theory for calculating the properties of shock and expansion waves in supersonic flow in 1908 (two chapters in this book). As a byproduct they produced the theory for *oblique shock*. In 1925 Prandtl became the director of the Kaiser Wilhelm Institute for Flow Investigation at Göttingen. By the 1930s, he was known worldwide as the leader in the science of fluid dynamics. Prandtl also contributed to research in many areas, such as meteorology and structural mechanics.

Ludwig Prandtl worked at Göttingen until his death on August 15, 1953. His work and achievements in fluid dynamics resulted in equations that simplified understanding, and many are still used today. Therefore many referred to him as the father of modern fluid mechanics. Ludwig Prandtl died in Göttingen, Germany on August 15th 1953.

Prandtl's other contributions include: the introduction of the Prandtl number in fluid mechanics, airfoils and wing theory (including theories of aerodynamic interference, wing-fuselage, wing-propeller, biplane, etc); fundamental studies in the wind

tunnel, high speed flow (correction formula for subsonic compressible flows), theory of turbulence. His name is linked to the following:

- Prandtl number (heat transfer problems)
- Prandtl-Glauert compressibility correction
- Prandtl's boundary layer equation
- Prandtl's lifting line theory
- Prandtl's law of friction for smooth pipes
- Prandtl-Meyer expansion fans (supersonic flow)
- Prandtl's Mixing Length Concept (theory of turbulence)

Theodor Meyer

Meyer[52] was Prandtl's student who in one dissertation was able to build large part of base of the modern compressible flow. The two chapters in this book, Prandtl–Meyer flow and oblique shock are directly based on his ideas. Settles et al in their paper argues that this thesis is the most influential dissertation in the entire field of fluid mechanics. No matter if you accept this opinion, it must be the most fundamental thesis or work in the field of compressible flow (about 20.08% page wise of this book.).

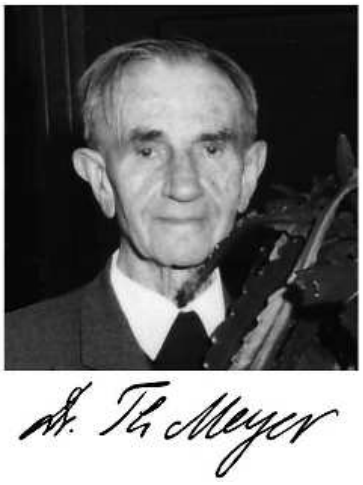


Fig. -1.12. Thedor Meyer's photo.

One of the questions that one can ask, what is about Meyer's education that brought this success. In his family, he was described as math genius who astonished his surroundings. What is so striking is the list of his instructors who include Frobenius (Group theory), Helmert (theory of errors), Hettner (chorology), Knoblauch , Lehmann-Filhes (orbit of double star), Edmund Landau (number theory), F. Schottkyand (elliptic, abelian, and theta functions and invented Schottky groups), mathematicians Caratheodory (calculus of variations, and measure theory), Herglotz (seismology), Hilbert, Klein, Lexis, Runge (Runge–Kutta method) and Zermelo (axiomatic set theory), Abraham (electron), Minkowski (mathematical base for

[52]This author is grateful to Dr. Settles and colleagues who wrote a very informative article about Meyer as a 100 years anniversary to his thesis. The material in this section was taken from Settles, G. S.,et al. "Theodor Meyer–Lost pioneer of gas dynamics" Prog. Aero space Sci(2009), doi:10.1016 j. paerosci.2009.06.001. More information can be found in that article.

theory of relativity), Prandtl, and more. This list demonstrates that Meyer had the best education one can have at the turn of century. It also suggest that moving between good universities (3 universities) is a good way to absorb knowledge and good research skills. This kind of education provided Meyer with the tools to tackle the tough job of compressible flow.

What is interesting about his work is that Mach number concept was not clear at that stage. Thus, the calculations (many hand numerical calculations) were complicated by this fact which further can magnify his achievement. Even though the calculations where carried out in a narrow range. Meyer's thesis is only 46 pages long but it include experimental evidence to prove his theory in Prandtl–Meyer function and oblique shock. According to Settles, this work utilized Schlieren images getting quantitative measurements probably for the first time. Meyer also was the first one to look at the ratio of the static properties to the stagnation proprieties[53].

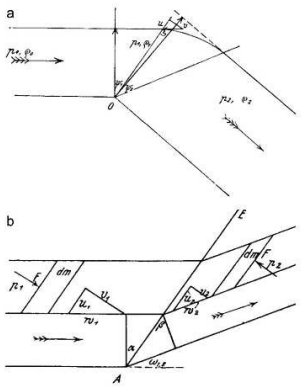


Fig. -1.13. The diagram is taken from Meyer's dissertation showing the schematic of oblique shock and the schematic of Prandtl–Meyer fan.

Ackeret attributed the oblique shock theory to Meyer but later this attribution was dropped and very few books attribute this Meyer (or even Prandtl). Among the very few who got this right is this book. The name Prandtl–Meyer is used because some believe that Prandtl conceived the concept and let Meyer to do the actual work. This contribution is to the mythical Prandtl ability to "solve" equations without doing the math. However, it is not clear that Prandtl indeed conceived or dealt with this issues besides reviewing Meyer ideas. What it is clear that the rigor mathematics is of Meyers and physical intuition of Prandtl were present. There is also a question of who came out with the "method of characteristics," Prandtl or Meyer.

Meyer was the first individual to use the shock polar diagram. Due to his diagram, he was able to see the existence of the weak and strong shock. Only in 1950, Thomson was able to see the third shock. It is not clear why Meyer missed the third root. Perhaps, it was Prandtl influence because he saw only two solutions in the experimental setting thus suggesting that only two solutions exists. This suggestion perhaps provides an additional indication that Prandtl was involved heavily in Meyer's thesis. Meyer also noticed that the deflection angle has a maximum.

Meyer was born to Theodor Meyer (the same name as his father) in July $1^{\underline{st}}$, 1882, and die March $8^{\underline{th}}$, 1972. Like Fanno, Meyer never was recognized for his contributions to fluid mechanics. During the years after Second World War, he was aware that his thesis became a standard material in every university in world. However, he never told

[53]This issue is considered still open by this author. It is not clear who use first and coin the term stagnation properties.

his great achievements to his neighbors or his family or colleagues in the high school where he was teaching. One can only wonder why this field rejects very talented people. Meyer used the symbol v which is still used to this today for the function.

E.R.G. Eckert

Fig. -1.14. The photo of Ernst Rudolf George Eckert with Bar-Meir's family Aug 1997

Eckert was born in September 13, 1904 in Prague, where he studied at the German Institute of Technology. He received his engineer deploma in 1927, and defend his (engineering sciences) Ph.D. in 1931. He work mostly on radtion for a while in the same pace where he studied. He went to work with Schmidt in Danzig known for experimatal experts in exsact messurement. That was the time that he develop the understading dimensional analysis in the heat transfer in particular and fluid mechanics in general education must be taught. He was critized for using and teaching dimensional analysis. During World War II, he developed methods for jet engine turbine blade cooling so they wouldn't burn up in space. He emigrated to the United States after the war, and served as a consultant to the U.S. Air Force and the National Advisory Committee for Aeronautics before coming to Minnesota.

Eckert developed the understanding of heat dissipation in relation to kinetic energy, especially in compressible flow. Hence, the dimensionless group has been designated as the Eckert number, which is associated with the Mach number. Schlichting suggested this dimensionless group in honor of Eckert. In addition to being named to the National Academy of Engineering in 1970, he authored more than 500 articles and received several medals for his contributions to science. His book "Introduction to the Transfer of Heat and Mass," published in 1937, is still considered a fundamental text in the field.

Eckert was an excellent mentor to many researchers (including this author), and he had a reputation for being warm and kindly. He was also a leading figure in bringing together engineering in the East and West during the Cold War years.

Ascher Shapiro

MIT Professor Ascher Shapiro[54], the Eckert equivalent for the compressible flow, was instrumental in using his two volume book "The Dynamics of Thermodynamics of the Compressible Fluid Flow," to transform the gas dynamics field to a coherent text material for engineers. Furthermore, Shapiro's knowledge of fluid mechanics enabled him to "sew" the missing parts of the Fanno line with Moody's diagram to create the most useful model in compressible flow. While Shapiro viewed gas dynamics mostly

[54]Parts taken from Sasha Brown, MIT

through aeronautic eyes, the undersigned believes that Shapiro was the first one to propose an isothermal flow model that is not part of the aeronautic field. Therefore it is being proposed to call this model Shapiro's Flow.

In his first 25 years Shapiro focused primarily on power production, high-speed flight, turbomachinery and propulsion by jet engines and rockets. Unfortunately for the field of Gas Dynamics, Shapiro moved to the field of biomedical engineering where he was able to pioneer new work. Shapiro was instrumental in the treatment of blood clots, asthma, emphysema and glaucoma.

Shapiro grew up in New York City and received his S.B. in 1938 and the Sc.D. (It is M.I.T.'s equivalent of a Ph.D. degree) in 1946 in mechanical engineering from MIT. He was assistant professor in 1943, three years before receiving his Sc.D. In 1965 he became the Department of Mechanical Engineering head until 1974. Shapiro spent most of his active years at MIT. Ascher Shapiro passed way in November 2004.

Von Karman, Theodor

A brilliant scientist who was instrumental in constructing many of the equations and building the American aviation and space exploration. Von Karman studied fluid mechanics under Prandtl and during that time he saw another graduate student that was attempting to build "stable" experiment what will not have the vortexes. Von Karman recognized that this situation is inherently unstable and explained the scientific reasons for this phenomenon. Now this phenomenon known as Von Karman street. Among his achievement that every student study fluid mechanics is the development of the integral equation of boundary layer. Von Karman, a descendant of a famous Rabi Rabbi Judah Loew ben Bezalel (HaMaharl) was born raised in Hungary. Later he move to Germany to study under Prandtl. After his graduation he taught at Gottingen and later as director of the Aeronautical Institute at RWTH Aachen. As a Jew realized that he has to leave Germany during the raise of the Nazi. At 1930 he received offer and accept the directorship of a Laboratory at the California Institute of Technology.

His achievement in the area of compressible flow area focused around supersonic and rocketry. For example, he formulate the slender body equations to describe the fluid field around rockets. Any modern air plane exhibits the swept–back wings the Von Karman was instrumental in recognizing its importance. He construct with his student the Von Karman–Tsien compressibility correction. The Karman–Tsien compressibility correction is a nonlinear approximation for Mach number effects which works quite well when the velocities are subsonic. This expression relates the incompressible values to those in compressible flow. As his adviser, he left many students which have continued his legacy like Tsien who build the Chinese missile industry.

Zeldovich, Yakov Borisovich

"Before I meet you here I had thought, that you are a collective of authors, as Burbaki" Stephen W. Hawking.

The statement of Hawking perhaps can illustrate a prolific physicist born in Minsk. He played an important role in the development of Soviet nuclear and thermonuclear

weapons. His main contribution in the area compressible flow centered around the shock wave and detonation and material related to cosmotology. Zeldovich develop several reatlition for the limiting cases still in use today. For example he developed the ZND detonation model (where the leter "Z" stands for Zeldovich).

CHAPTER 2

Review of Thermodynamics

In this chapter, a review of several definitions of common thermodynamics terms is presented. This introduction is provided to bring the student back to current place with the material.

2.1 Basic Definitions

The following basic definitions are common to thermodynamics and will be used in this book.

Work

In mechanics, the work was defined as

$$\text{mechanical work} = \int \mathbf{F} \bullet \mathbf{d}\ell = \int P dV \tag{2.1}$$

This definition can be expanded to include two issues. The first issue that must be addressed, that work done on the surroundings by the system boundaries similarly is positive. Two, there is a transfer of energy so that its effect can cause work. It must be noted that electrical current is a work while heat transfer isn't.

System

This term will be used in this book and it is defined as a continuous (at least partially) fixed quantity of matter. The dimensions of this material can be changed. In this definition, it is assumed that the system speed is significantly lower than that of the speed of light. So, the mass can be assumed constant even though the true conservation law applied to the combination of mass energy (see Einstein's law). In fact for almost all engineering purpose this law is reduced to two separate laws of mass conservation and energy conservation.

Our system can receive energy, work, etc as long the mass remain constant the definition is not broken.

Thermodynamics First Law

This law refers to conservation of energy in a non accelerating system. Since all the systems can be calculated in a non accelerating systems, the conservation is applied to all systems. The statement describing the law is the following.

$$Q_{12} - W_{12} = E_2 - E_1 \tag{2.2}$$

The system energy is a state property. From the first law it directly implies that for process without heat transfer (adiabatic process) the following is true

$$W_{12} = E_1 - E_2 \tag{2.3}$$

Interesting results of equation (2.3) is that the way the work is done and/or intermediate states are irrelevant to final results. There are several definitions/separations of the kind of works and they include kinetic energy, potential energy (gravity), chemical potential, and electrical energy, etc. The internal energy is the energy that depends on the other properties of the system. For example for pure/homogeneous and simple gases it depends on two properties like temperature and pressure. The internal energy is denoted in this book as E_U and it will be treated as a state property.

The potential energy of the system is depended on the body force. A common body force is the gravity. For such body force, the potential energy is mgz where g is the gravity force (acceleration), m is the mass and the z is the vertical height from a datum. The kinetic energy is

$$K.E. = \frac{mU^2}{2} \tag{2.4}$$

Thus, the energy equation can be written as

System Energy Conservation

$$\frac{m\,{U_1}^2}{2} + m\,g\,z_1 + E_{U1} + Q = \frac{m\,{U_2}^2}{2} + m\,g\,z_2 + E_{U2} + W \tag{2.5}$$

For the unit mass of the system equation (2.5) is transformed into

System Energy Conservation per Unit

$$\frac{{U_1}^2}{2} + gz_1 + E_{u1} + q = \frac{{U_2}^2}{2} + gz_2 + E_{u2} + w \tag{2.6}$$

where q is the energy per unit mass and w is the work per unit mass. The "new" internal energy, E_u, is the internal energy per unit mass.

Since the above equations are true between arbitrary points, choosing any point in time will make it correct. Thus differentiating the energy equation with respect to time yields the rate of change energy equation. The rate of change of the energy transfer is

$$\frac{DQ}{Dt} = \dot{Q} \tag{2.7}$$

In the same manner, the work change rate transferred through the boundaries of the system is

$$\frac{DW}{Dt} = \dot{W} \tag{2.8}$$

Since the system is with a fixed mass, the rate energy equation is

$$\dot{Q} - \dot{W} = \frac{D\,E_U}{Dt} + mU\frac{DU}{Dt} + m\frac{D\,B_f\,z}{Dt} \tag{2.9}$$

For the case were the body force, B_f, is constant with time like in the case of gravity equation (2.9) reduced to

System Energy Conservation per Time

$$\dot{Q} - \dot{W} = \frac{D\,E_U}{Dt} + m\,U\frac{DU}{Dt} + m\,g\frac{D\,z}{Dt} \tag{2.10}$$

The time derivative operator, D/Dt is used instead of the common notation because it referred to system property derivative.

Thermodynamics Second Law

There are several definitions of the second law. No matter which definition is used to describe the second law it will end in a mathematical form. The most common mathematical form is Clausius inequality which state that

$$\oint \frac{\delta Q}{T} \geq 0 \tag{2.11}$$

The integration symbol with the circle represent integral of cycle (therefor circle) in with system return to the same condition. If there is no lost, it is referred as a reversible process and the inequality change to equality.

$$\oint \frac{\delta Q}{T} = 0 \tag{2.12}$$

The last integral can go though several states. These states are independent of the path the system goes through. Hence, the integral is independent of the path. This observation leads to the definition of entropy and designated as S and the derivative of entropy is

$$ds \equiv \left(\frac{\delta Q}{T}\right)_{\text{rev}} \tag{2.13}$$

Performing integration between two states results in

$$S_2 - S_1 = \int_1^2 \left(\frac{\delta Q}{T}\right)_{\text{rev}} = \int_1^2 dS \tag{2.14}$$

One of the conclusions that can be drawn from this analysis is for reversible and adiabatic process $dS = 0$. Thus, the process in which it is reversible and adiabatic, the entropy remains constant and referred to as It can be noted that there is a possibility that a process can be irreversible and the right amount of heat transfer to have zero change entropy change. Thus, the reverse conclusion that zero change of entropy leads to reversible process, isn't correct.

For reversible process equation (2.12) can be written as

$$\delta Q = T\, dS \tag{2.15}$$

and the work that the system is doing on the surroundings is

$$\delta W = P\, dV \tag{2.16}$$

Substituting equations (2.15) (2.16) into (2.10) results in

$$T dS = d\, E_U + P\, dV \tag{2.17}$$

Even though the derivation of the above equations were done assuming that there is no change of kinetic or potential energy, it still remain valid for all situations. Furthermore, it can be shown that it is valid for reversible and irreversible processes.

Enthalpy

It is a common practice to define a new property, which is the combination of already defined properties, the enthalpy of the system.

$$H = E_U + P\, V \tag{2.18}$$

The specific enthalpy is enthalpy per unit mass and denoted as, h.

Or in a differential form as

$$dH = dE_U + dP\, V + P\, dV \tag{2.19}$$

Combining equations (2.18) the (2.17) yields

Fundamental Entropy

$$T\, dS = dH - V\, dP \tag{2.20}$$

For isentropic process, equation (2.17) is reduced to $dH = VdP$. The equation (2.17) in mass unit is

$$T\, ds = du + P\, dv = dh - \frac{dP}{\rho} \tag{2.21}$$

when the density enters through the relationship of $\rho = 1/v$.

Specific Heats

The change of internal energy and enthalpy requires new definitions. The first change of the internal energy and it is defined as the following

Volume Specific Heat

$$C_v \equiv \left(\frac{\partial E_u}{\partial T}\right) \tag{2.22}$$

And since the change of the enthalpy involve some kind of work is defined as

Pressure Specific Heat

$$C_p \equiv \left(\frac{\partial h}{\partial T}\right) \tag{2.23}$$

The ratio between the specific pressure heat and the specific volume heat is called the ratio of the specific heat and it is denoted as, k.

Specific Heat Ratio

$$k \equiv \frac{C_p}{C_v} \tag{2.24}$$

For solid, the ratio of the specific heats is almost 1 and therefore the difference between them is almost zero. Commonly the difference for solid is ignored and both are assumed to be the same and therefore referred as C. This approximation less strong for liquid but not by that much and in most cases it applied to the calculations. The ratio the specific heat of gases is larger than one.

Equation of State

Equation of state is a relation between state variables. Normally the relationship of temperature, pressure, and specific volume define the equation of state for gases. The simplest equation of state referred to as ideal gas. and it is defined as

$$P = \rho\, R\, T \tag{2.25}$$

Application of Avogadro's law, that "all gases at the same pressures and temperatures have the same number of molecules per unit of volume," allows the calculation of a "universal gas constant." This constant to match the standard units results in

$$\bar{R} = 8.3145 \frac{kj}{kmol\ K} \tag{2.26}$$

Thus, the specific gas can be calculate as

$$R = \frac{\bar{R}}{M} \tag{2.27}$$

The specific constants for select gas at 300K is provided in table 2.1.

Table -2.1. Properties of Various Ideal Gases at [300K]

Gas	**Chemical Formula**	Molecular Weight	$\mathbf{R}\ \left[\frac{kJ}{Kg\,K}\right]$	$\mathbf{C_P}\ \left[\frac{kJ}{KgK}\right]$	$\mathbf{C_v}\ \left[\frac{kJ}{KgK}\right]$	**k**
Air	-	28.970	0.28700	**1.0035**	0.7165	1.400
Argon	**Ar**	39.948	0.20813	**0.5203**	0.3122	1.667
Butane	C_4H_{10}	58.124	0.14304	**1.7164**	1.5734	1.091
Carbon Dioxide	CO_2	44.01	0.18892	**0.8418**	0.6529	1.289
Carbon Monoxide	CO	28.01	0.29683	**1.0413**	0.7445	1.400
Ethane	C_2H_6	30.07	0.27650	**1.7662**	1.4897	1.186
Ethylene	C_2H_4	28.054	0.29637	**1.5482**	1.2518	1.237
Helium	He	4.003	2.07703	**5.1926**	3.1156	1.667
Hydrogen	H_2	2.016	4.12418	**14.2091**	10.0849	1.409
Methane	CH_4	16.04	0.51835	**2.2537**	1.7354	1.299
Neon	Ne	20.183	0.41195	**1.0299**	0.6179	1.667
Nitrogen	N_2	28.013	0.29680	**1.0416**	0.7448	1.400
Octane	C_8H_{18}	114.230	0.07279	**1.7113**	1.6385	1.044
Oxygen	O_2	31.999	0.25983	**0.9216**	0.6618	1.393
Propane	C_3H_8	44.097	0.18855	**1.6794**	1.4909	1.126
Steam	H_2O	18.015	0.46153	**1.8723**	1.4108	1.327

From equation of state (2.25) for perfect gas, it follows

$$d(P\,v) = R\,dT \tag{2.28}$$

For perfect gas

$$dh = dE_u + d(Pv) = dE_u + d(RT) = f(T) \text{ (only)} \tag{2.29}$$

From the definition of enthalpy it follows that

$$d(Pv) = dh - dE_u \tag{2.30}$$

Utilizing equation (2.28) and subsisting into equation (2.30) and dividing by dT yields

$$C_p - C_v = R \tag{2.31}$$

This relationship is valid only for ideal/perfect gases.

The ratio of the specific heats can be expressed in several forms as

C_v =f(R)

$$C_v = \frac{R}{k-1} \tag{2.32}$$

and

C_p =f(R)

$$C_p = \frac{k\,R}{k-1} \tag{2.33}$$

The specific heat ratio, k value ranges from unity to about 1.667. These values depend on the molecular degrees of freedom (more explanation can be obtained in Van Wylen "F. of Classical thermodynamics." The values of several gases can be approximated as ideal gas and are provided in Table (2.1).

The entropy for ideal gas can be simplified as the following

$$s_2 - s_1 = \int_1^2 \left(\frac{dh}{T} - \frac{dP}{\rho T} \right) \tag{2.34}$$

Using the identities developed so far one can find that

$$s_2 - s_1 = \int_1^2 C_p \frac{dT}{T} - \int_1^2 \frac{R\,dP}{P} = C_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1} \tag{2.35}$$

Or using specific heat ratio equation (2.35) transformed into

$$\frac{s_2 - s_1}{R} = \frac{k}{k-1} \ln \frac{T_2}{T_1} - \ln \frac{P_2}{P_1} \tag{2.36}$$

For isentropic process, $\Delta s = 0$, the following is obtained

$$\ln \frac{T_2}{T_1} = \ln \left(\frac{P_2}{P_1} \right)^{\frac{k-1}{k}} \tag{2.37}$$

There are several famous identities that results from equation (2.37) as

Isentropic Relationship

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &= \left(\frac{T_2}{T_1}\right)^{\frac{1}{k-1}} = \left(\frac{P_2}{P_1}\right)^{\frac{1}{k}} = \left(\frac{V_1}{V_2}\right) \\
\frac{T_2}{T_1} &= \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} = \left(\frac{\rho_2}{\rho_1}\right)^{k-1} = \left(\frac{V_1}{V_2}\right)^{(k-1)} \\
\frac{P_2}{P_1} &= \left(\frac{T_2}{T_1}\right)^{\frac{k}{k-1}} = \left(\frac{\rho_2}{\rho_1}\right)^{k} = \left(\frac{V_1}{V_2}\right)^{k} \\
\frac{V_2}{V_1} &= \left(\frac{T_1}{T_2}\right)^{\frac{1}{k-1}} = \left(\frac{\rho_1}{\rho_2}\right) = \left(\frac{P_1}{P_2}\right)^{\frac{1}{k}}
\end{aligned} \tag{2.38}$$

The ideal gas model is a simplified version of the real behavior of real gas. The real gas has a correction factor to account for the deviations from the ideal gas model. This correction factor referred as the compressibility factor and defined as

Compressibility Factor

$$Z = \frac{V_{\text{actual}}}{V_{\text{ideal gas}}} = \frac{P\,v}{R\,T} = \frac{P}{\rho\,R\,T} \tag{2.39}$$

One of the common way to estimate the compressibility factor (is by using or based on Redlick-Kwong Equation. In this method the equation of state is

$$P = \frac{R\,T}{V_m - B} - \frac{A}{\sqrt{T}\,V_m\,(V_m + B)} \tag{2.40}$$

where the V_m is the molar volume, A is a coefficient accounting for attractive potential of molecules, and B is a coefficient that accounting for volume correction.

The coefficients are a function of gas. These coefficients can be estimated using the critical point of the gas

$$A = \frac{0.4275\,R^2\,{T_c}^{2.5}}{P_c}, \qquad B = \frac{0.08664\,R\,T_c}{P_c} \tag{2.41}$$

where: T_c is the critical temperature, and P_c is the critical pressure.

Expressing[1] the volume as a faction of the other parameters in equation (2.39) and then substituting into equation (2.40) transfomred it into cubic equation of Z as

$$Z^3 - Z^2 - \eta_1\,Z - \eta_2 = 0 \tag{2.42}$$

[1] This idea was suggested by Cultip and Shacham

where

$$\phi_1 = 0.42747 \left(\frac{P_r}{T_r{}^{\frac{5}{2}}} \right) \tag{2.43}$$

$$\phi_2 = 0.08664 \left(\frac{P_r}{T_r} \right) \tag{2.44}$$

$$\eta_1 = {\phi_2}^2 + \phi_2 - \phi_1 \qquad \eta_2 = \phi_1 \, \phi_2 \tag{2.45}$$

Z can be solve analytically and will be presented in a Figure.

2.1.1 Maximum Work of Expansion Process

In some industries, such as the printing, there is a need to obtain a high temperature for a short time (to back the paper) and later the paper has to be cooled to a room temperature. On the other hand, when one want to absorb the car energy during the stopping process and reuse it later leads to a theoretical question. This question what is the minimum work to compress substance or maximum work or energy that can be obtained from substance undergoes change of the condition.. The compressibility has a significant effect on the work that can be obtained or required in a piston cylinder configuration and others configurations. For example, a shock tube is a device which used to obtain high temperature to study explosions and potentially has civilian applications. What is the minimum energy or work which required to compress a piston from state $\mathcal{A}$ to state $\mathcal{B}$. Clearly, the compression has to occur in an isentropic process. The equation of state has to be specified to have a specific equation describing the situation. The simplest equation is the ideal gas model. The general case has three dimensions and requires a complex mathematical treatment. To simplify the discussion, it is assumed that the process occurs in an one dimensional case.

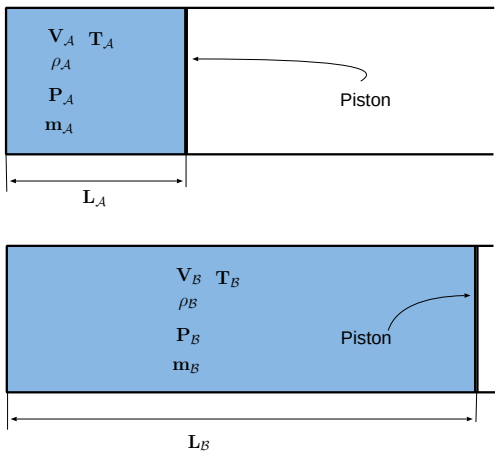


Fig. -2.1. Cylinder and piston configuration of maximum work.

Perfect Gas

The gas confined by a cylinder and piston undergoes isentropic process during expansion. First the analysis will be discussed for the case of perfect gas. Every stage of the process is denoted by x and $L_{\mathcal{B}} > x > L_{\mathcal{A}}$. The mass conservation ($m_{\mathcal{A}} = m(x)$)

and assuming perfect gas model is

$$\frac{P_{\mathcal{A}}\,V_{\mathcal{A}}}{T_{\mathcal{A}}\cancel{R}} = \frac{P(x)\,V(x)}{T(x)\cancel{R}} \Longrightarrow P(x) = P_{\mathcal{A}}\,\frac{T(x)}{T_{\mathcal{A}}}\,\frac{\overbrace{\cancel{A}L_{\mathcal{A}}}^{V_{\mathcal{A}}}}{\underbrace{V(x)}_{\cancel{A}(L_{\mathcal{A}}+x)}} = P_{\mathcal{A}}\,\frac{T(x)\,L_{\mathcal{A}}}{T_{\mathcal{A}}\,(L_{\mathcal{A}}+x)} \tag{2.46}$$

Using the isentropic relationship during the process see equation table (2.38) apply,

$$P(x) = P_{\mathcal{A}}\left(\frac{V_{\mathcal{A}}}{V(x)}\right)^{k} \tag{2.47}$$

It can be noticed that the area can be canceled (see equation (2.46) for similar explanation) and the isentropic relationship can be described by

$$P_(x) = P_{\mathcal{A}}\left(\frac{L_{\mathcal{A}}}{L_{\mathcal{A}}+x}\right)^{k} \tag{2.48}$$

Dimensional analysis[2] can be used to minimize the labor and introducing a dimensionless variable

$$\xi = x/L_{\mathcal{A}} \tag{2.49}$$

By introducing this variable, it is acknowledged that this dimensionless parameter is effecting the solution of the problem. The work done by the gas on the piston is

$$W_{\mathcal{A}\to\mathcal{B}} = \int_0^{L_{\mathcal{B}}-L_{\mathcal{A}}} \overbrace{A\,P(x)}^{F(x)}\,dx \tag{2.50}$$

Equation (2.50) by area, A, which is a constant and substituting equation (2.48) yields

$$\frac{W_{\mathcal{A}\to\mathcal{B}}}{A} = \int_0^{L_{\mathcal{B}}-L_{\mathcal{A}}} P_{\mathcal{A}1}\left(\frac{L_{\mathcal{A}}}{(L_{\mathcal{A}}+x)}\right)^{k}\frac{L_{\mathcal{A}}\,dx}{L_{\mathcal{A}}} \tag{2.51}$$

Equation (2.51) can be transformed into a dimensionless form when dividing by Pressure and length. Also notice the integration limits change as following

$$\frac{W_{\mathcal{A}\to\mathcal{B}}}{A\,L_{\mathcal{A}}\,P_{\mathcal{A}}} = \int_0^{L_{\mathcal{B}}-L_{\mathcal{A}}}\left(\frac{\cancel{L_{\mathcal{A}}}(1)}{\cancel{L_{\mathcal{A}}}(1+\xi)}\right)^{k} d\xi = \int_0^{\frac{L_{\mathcal{B}}}{L_{\mathcal{A}}}-1}\left(\frac{1}{1+\xi}\right)^{k} d\xi \tag{2.52}$$

Carrying the integration of equation (2.52) yields

$$\frac{W_{\mathcal{A}\to\mathcal{B}}}{A\,L\,P_{\mathcal{A}1}} = \left.\frac{(1+\xi)^{1-k}}{1-k}\right|_0^{\frac{L_{\mathcal{B}}}{L_{\mathcal{A}}}-1} \tag{2.53}$$

$$\frac{W_{\mathcal{A}\to\mathcal{B}}}{A\,L\,P_{\mathcal{A}1}} = \frac{1}{k-1}\left[1-\left(\frac{L_{\mathcal{B}}}{L_{\mathcal{A}}}\right)^{1-k}\right] \tag{2.54}$$

[2] "Basics of Fluid Mechanics" by Bar-Meir has extensive discussion about this point.

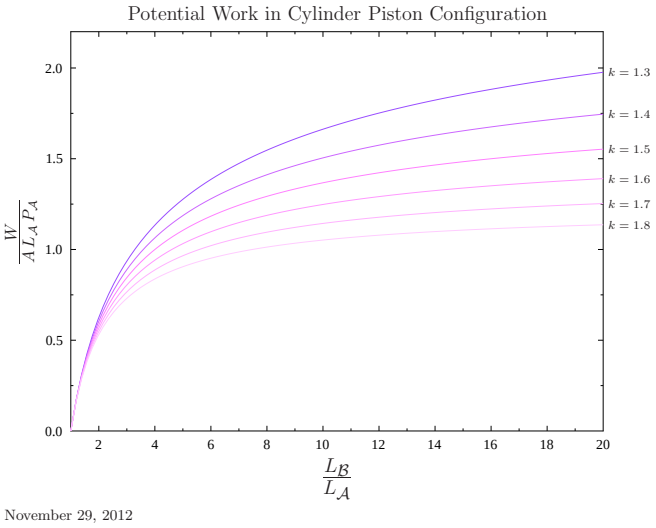


Fig. -2.2. Dimensionless work available in a cylinder piston configuration.

2.2 The Velocity–Temperature Diagram

The velocity–temperature (U–T) diagram was developed by Stodola (1934) and expended by Spalding (1954). In the U–T diagram, the logarithms of temperature is plotted as a function of the logarithms of velocity. For simplicity, the diagram here deals with perfect gas only (constant specific heat)[3]. The ideal gas equation (2.25) was described before. This diagram provides a graphical way to analysis the flow and to study the compressible flow because two properties defines the state.

The enthalpy is a linear function of the temperature due to the assumptions employed here (the pressure does not affect the enthalpy). The energy equation (2.18) can be written for adiabatic process as

$$h + \frac{U^2}{2} = constant_1 \tag{2.55}$$

Taking the logarithms of both sides of equation (2.55) results in

$$\log\left(h + \frac{U^2}{2}\right) = constant_2 \tag{2.56}$$

or

$$\log\left(T + \frac{U^2}{2\,C_p}\right) = constant_3 \tag{2.57}$$

[3]The perfect gas model is used because it provides also the trends of more complicated model.

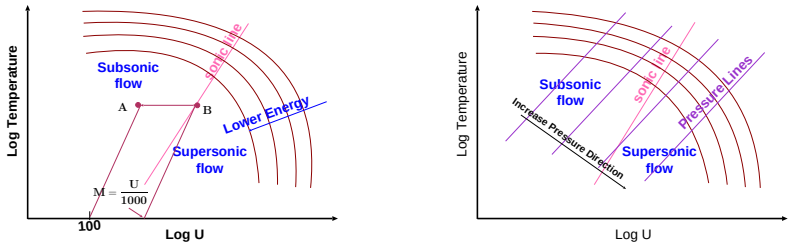
Example 2.1:
Determine the relationship between $constant_3$ *in equation* (2.57) *to* $constant_1$ *in equation 2.55.*

SOLUTION

Under construction

End Solution

From equation (2.56), it can be observed as long as the velocity square is relatively small compared to multiplication of the specific heat by the temperature, it remains close to constant. Around $U^2 = 2\,C_p\,T$ the velocity drops rapidly. These lines are referred to as energy lines because kinetic energy and thermal remain constant. These lines are drawn in Figure 2.3(b).



(a) The pressure lines

(b) Various lines in Velocity–Temperature diagrams

Fig. -2.3. caption

The sonic line (the speed of sound will be discussed in Chapter 4) is a line that given by the following equation

$$U = c = \sqrt{kRT} \rightarrow \ln c = \frac{1}{2}\log\left(kRT\right) \tag{2.58}$$

The reason that logarithms scales are used is so that the relative speed (U/c also known as Mach number, will be discussed page 86) for any point on the diagram, can be directly measured. For example, the Mach number of point $\mathbf{A}$, shown in Figure 2.3(a), is obtained by measuring the distance $\mathbf{A} - \mathbf{B}$. The distance $\mathbf{A} - \mathbf{B}$ represent the ratio of the speed of sound because

$$\mathbf{A} - \mathbf{B} = \log U|_A - \log c|_B = \log \frac{U|_A}{c} \tag{2.59}$$

For example, when copying the distance $\mathbf{A} - \mathbf{B}$ to the logarithms scale results in Mach number. For instance, copying the distance to starting point of 100, the Mach number at point $\mathbf{A}$ will be the read number from the scale divided by 1000.

Mass conservation reads

$$\frac{\dot{m}}{A} = U\,\rho \tag{2.60}$$

Subsisting the equation of state (2.25) into equation (2.60) results in

$$\frac{\dot{m}\,R}{A\,P} = \frac{U}{T} \tag{2.61}$$

Taking the logarithms from both sides results in

$$\log\left(\frac{\dot{m}\,R}{A\,P}\right) = \log\left(\frac{U}{T}\right) \tag{2.62}$$

After rearrangement of equation (2.62) obtain

$$\log\left(\frac{\dot{m}\,R}{A\,P}\right) = \log U - \log T \tag{2.63}$$

or

$$\log T = \log U - \log\left(\frac{\dot{m}\,R}{A\,P}\right) \tag{2.64}$$

Figure 2.3(b) depicts these lines which referred to as the pressure (mass flow rate) lines. For constant mass flow and pressure, $\log\,T$ is linearly depend on $\log\,U$. In fact, for constant value of $\log\frac{\dot{m}\,R}{A\,P}$ the pressure line is at 45° on diagram.

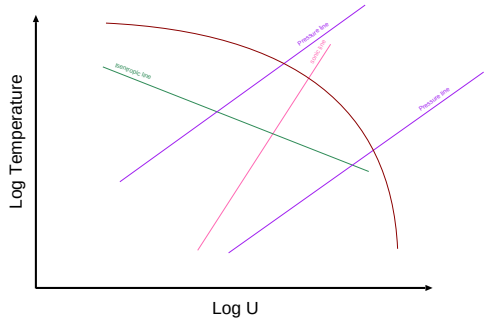


Fig. -2.4. The $\ln$ *temperature versus of the velocity diagram*

The constant momentum can be written as

$$P + \frac{U^2}{\rho} = constant = P_0 \tag{2.65}$$

Where P_0 is the pressure if the velocity was zero. It can be observed that from perfect gas model and continuing equation the following is obtained

$$P = \frac{\dot{m}\, R\, T}{U\, A} \tag{2.66}$$

Utilizing the perfect gas state equation and equation (2.66) and substituting into equation (2.65) yields

$$\frac{\dot{m}\, R\, T}{U\, A} + \frac{\dot{m}\, U^2}{A\, U} = P_0 \tag{2.67}$$

Or in simplified form

$$T = -\frac{U^2}{2\, R} + \frac{P_0\, A\, U}{\dot{m}\, R} \tag{2.68}$$

The temperature is upside down parabola in relationship to velocity on the momentum lines.

$$\log T = \log\left(-\frac{U^2}{2\, R} + \frac{P_0\, A\, U}{\dot{m}\, R}\right) \tag{2.69}$$

These line also called Stodola lines or Rayleigh lines.

The maximum of the temperature on the momentum line can be calculate by taking the derivative and equating to zero.

$$\frac{dT}{dU} = -\frac{2\, U}{2\, R} + \frac{P_0\, A}{\dot{m}\, R} = \tag{2.70}$$

The maximum temperature is then

$$U = \frac{P_0\, A}{\dot{m}} \tag{2.71}$$

It can be shown that this velocity is related to $\sqrt{k}T_0$ where the T_0 is the velocity zero.

CHAPTER 3

Basic of Fluid Mechanics

3.1 Introduction

The reader is expected to be familiar with the fundamentals of fluid mechanics and this review is provided as refreshment. These basic principles and concepts are to be use in the book and are a building blocks of the understanding the material presented later. Several concepts are reviewed such as control volume. Several applications of the fluid mechanics will demonstrated. First, a discussion about fluid proprieties (related to compressible flow) is presented. The integral and differential methods are described. Later, a discussion about the governing equations in fluid mechanics is presented.

3.2 Fluid Properties

3.2.1 Kinds of Fluids

Some differentiate fluids from solid by the reaction to shear stress. Generally it is accepted that the fluid continuously and permanently deformed under shear stress while solid exhibits a finite deformation which does not change with time. It is also said that the liquid cannot return to their original state after the deformation. This differentiation leads to three groups of materials: solids and fluids and those between these two limits. This test creates a new material group that shows dual behaviors; under certain limits; it behaves like solid and under others it behaves like fluid. This book deals with only clear fluid (at least, this is the intention at this stage). The fluid is mainly divided into two categories: liquids and gases. The main difference between the liquids and gases state is that gas will occupy the whole volume while liquids has an almost fix volume. This difference can be, for most practical purposes considered sharper.

Density

The density is the main property which causes the field of compressible flow. The density is a property which requires that the fluid to be continuous. The density can be changed and it is a function of time and space (location) but must be continues. It doesn't mean that a sharp and abrupt change in fields cannot occur. The continues requirement is referred to the fact that density is independent of the sampling size. After certain sampling size, the density remains constant. Thus, the density is defined as

$$\rho = \lim_{\Delta V \longrightarrow \varepsilon} \frac{\Delta m}{\Delta V} \tag{3.1}$$

It must be noted that ε is chosen so that the continuous assumption is not broken, that is, it did not reach/reduced to the size where the atoms or molecular statistical calculations are significant.

3.2.2 Viscosity

The shear stress is part of the pressure tensor. This book deals with Newtonian fluid and hence, applying the linear relationship can be written for the shear stress

$$\tau_{xy} = \mu \frac{dU}{dy} \tag{3.2}$$

Where μ is called the absolute viscosity or dynamic viscosity. Newtonian fluids are fluids which the ratio is constant. Many fluids fall into this category such as air, water etc. This approximation is appropriate for many other fluids but only within some ranges.

Equation (3.2) can be interpreted as momentum in the x direction transferred into the y direction. Thus, the viscosity is the resistance to the flow (flux) or the movement. The property of viscosity, which is exhibited by all fluids, is due to the existence of cohesion and interaction between fluid molecules. These cohesions and interactions hamper the flux in y–direction. Some referred to shear stress as viscous flux of x–momentum in the y–direction. The units of shear stress are the same as flux per time as following

$$\frac{F}{A}\left[\frac{kg\,m}{sec^2}\,\frac{1}{m^2}\right] = \frac{\dot{m}\,U}{A}\left[\frac{kg}{sec}\,\frac{m}{sec}\,\frac{1}{m^2}\right]$$

Thus, the notation of τ_{xy} is easier to understand and visualize. In fact, this interpretation is more suitable to explain the molecular mechanism of the viscosity. The units of absolute viscosity are $[N\,sec/m^2]$.

Viscosity varies widely with temperature. However, temperature variation has an opposite effect on the viscosities of liquids and gases. The difference is due to their fundamentally different mechanism creating viscosity characteristics. In gases, molecules are sparse and cohesion is negligible, while in the liquids, the molecules are more compact and cohesion is more dominate. Thus, in gases, the exchange of

momentum between layers brought as a result of molecular movement normal to the general direction of flow, and it resists the flow. This molecular activity is known to increase with temperature, thus, the viscosity of gases will increase with temperature. This reasoning is a result of the considerations of the kinetic theory. This theory indicates that gas viscosities vary directly with the square root of temperature. In liquids, the momentum exchange due to molecular movement is small compared to the cohesive forces between the molecules. Thus, the viscosity is primarily dependent on the magnitude of these cohesive forces. Since these forces decrease rapidly with increases of temperature, liquid viscosities decrease as temperature increases.

Well above the critical point (two phase dome), both phases are only a function of the temperature. On the liquid side below the critical point, the pressure has minor effect on the viscosity. It must be stress that the viscosity in the dome is meaningless. There is no such a thing of viscosity at 30% liquid. It simply depends on the structure of the flow see for more detail in "Basic of Fluid Mechamics, Bar–Meir" in the chapter on multi phase flow. Oils have the greatest increase of viscosity with pressure which is a good thing for many engineering purposes.

3.2.3 Kinematic Viscosity

The kinematic viscosity is another way to look at the viscosity. The reason for this new definition is that some experimental data are given in this form. These results also explained better using the new definition. The kinematic viscosity embraces both the viscosity and density properties of a fluid. The above equation shows that the dimensions of ν to be square meter per second, $[m^2/sec]$, which are acceleration units (a combination of kinematic terms). This fact explains the name "kinematic" viscosity. The kinematic viscosity is defined as

$$\nu = \frac{\mu}{\rho} \tag{3.3}$$

The gas density decreases with the temperature. However, The increase of the absolute viscosity with the temperature is enough to overcome the increase of density and thus, the kinematic viscosity also increase with the temperature for many materials.

3.2.4 Bulk Modulus

Similar to solids (hook's law), fluids have a property that describes the volume change as results of pressure change for constant temperature. It can be noted that this property is not the result of the equation of state but related to it. Bulk modulus is usually obtained from experimental or theoretical or semi theoretical methods.

The bulk modulus is defined as

$$B_T = -v \left(\frac{\partial P}{\partial v} \right)_T \tag{3.4}$$

Using the identity of $v = 1/\rho$ transfers equation (3.4) into

$$B_T = \rho \left(\frac{\partial P}{\partial \rho} \right)_T \tag{3.5}$$

The bulk modulus for several selected liquids is presented in Table 3.1.

Table -3.1. The bulk modulus for selected material with the critical temperature and pressure $na \longrightarrow$ not available and $nf \longrightarrow$ not found (exist but was not found in the literature).

Chemical component	Bulk Modulus $10^9 \frac{N}{m}$	T_c	P_c
Acetic Acid	2.49	593K	57.8 [Bar]
Acetone	0.80	508 K	48 [Bar]
Benzene	1.10	562 K	4.74 [MPa]
Carbon Tetrachloride	1.32	556.4 K	4.49 [MPa]
Ethyl Alcohol	1.06	514 K	6.3 [Mpa]
Gasoline	1.3	nf	nf
Glycerol	4.03-4.52	850 K	7.5 [Bar]
Mercury	26.2-28.5	1750 K	172.00 [MPa]
Methyl Alcohol	0.97	Est 513	Est 78.5 [Bar]
Nitrobenzene	2.20	nf	nf
Olive Oil	1.60	nf	nf
Paraffin Oil	1.62	nf	nf
SAE 30 Oil	1.5	na	na
Seawater	2.34	na	na
Toluene	1.09	591.79 K	4.109 [MPa]
Turpentine	1.28	na	na
Water	2.15-2.174	647.096 K	22.064 [MPa]

Additional expansions for similar parameters are defined . The thermal expansion is defined as

$$\beta_P = \frac{1}{v} \left(\frac{\partial v}{\partial T} \right)_P \qquad \beta_v = \frac{1}{P} \left(\frac{\partial P}{\partial T} \right)_v \tag{3.6}$$

These parameters are related as

$$\beta_T = -\frac{\beta_v}{\beta_P} \tag{3.7}$$

The definition of bulk modulus will be used to calculate the the speed of sound in slightly compressed liquid.

3.3 The Control Volume and Mass Conservation

In this section the conservation of the mass, momentum, and energy equation are presented. In simple (solid) system, Newton second law is applied and is conserved because the object remains the same (no deformation). However, when the fluid system moves relative location of one particle to another is changing. Typically, one wants to find or to predict the velocities in the system. Thus, using the old approach requires to keep track of every particle (or small slabs). This kind of analysis is reasonable and it referred to in the literature as the Lagrangian Analysis. This name is in honored J. L. Langrange (1736–1813) who formulated the equations of motion for the moving fluid particles.

Even though the Lagrangian system looks reasonable, this system turned out to be difficult to solve and to analyze therefore it is used only in very few cases. The main difficulty lies in the fact that every particle has to be traced to its original state. Leonard Euler (1707–1783) suggested an alternative approach based on a defined volume. This methods is referred as Eulerian method. The Eulerian method focuses on a defined area or location to find the needed information. The use of the Eulerian methods leads to a set differentiation equations that is referred to as the Navier–Stokes equations which are commonly used. The Eulerian system leads to integral equations which will be used in several cases in this book.

3.3.1 Control Volume

The Eulerian method requires to define a control volume (sometime more than one). The control volume is a defined volume which is differentiated into two categories: non–deformable and deformable. Non–deformable control volume is a control volume which is fixed in space relatively to an one coordinate system. This coordinate system may be in a relative motion to another (almost absolute) coordinate system. Deformable control volume is a volume having part or all of its boundaries in motion during the process at hand. The control volume is used to build the conservation equations for the mass, momentum, energy, entropy etc. The choice of control volume (deformable or not) is a function to what bring a simpler solution.

3.3.2 Continuity Equation

The mass conservation of a system is

$$\frac{D\,m_{sys}}{Dt} = \frac{D}{Dt}\int_{V_{sys}} \rho dV = 0 \tag{3.8}$$

The system mass after some time is made of

$$m_{sys} = m_{c.v.} + m_{out} - m_{in} \tag{3.9}$$

Where m_{out} is the mass flow out and m_{in} is the mass flow in. The change with the time is zero and hence

$$0 = \frac{D\,m_{sys}}{Dt} = \frac{d\,m_{c.v.}}{dt} + \frac{d\,m_{out}}{dt} - \frac{d\,m_{in}}{dt} \tag{3.10}$$

The first term on the right hand side is converted to integral and the other two terms on the right hand side are combined and realizing that the sign can be accounted for flow in or out as

Continuity

$$\frac{d}{dt}\int_{c.v.} \rho_s dV = -\int_{S_{cv}} \rho\, U_{rn}\, dA \tag{3.11}$$

Equation (3.11) is essentially accounting of the mass that is the change is result of the in an out flow. The negative sign in surface integral is because flow out marked positive which reduces of the mass (negative derivative).

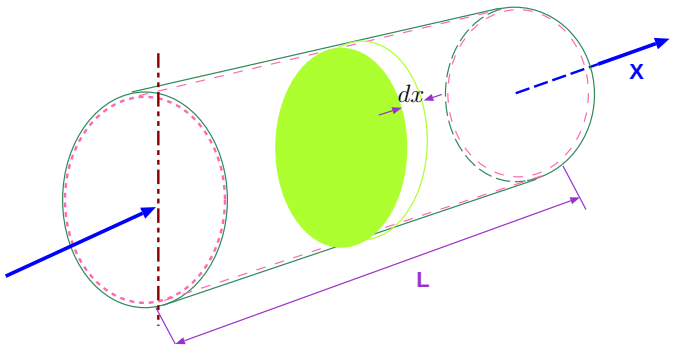


Fig. -3.1. Schematics of flow in in pipe with varying density as a function time for example 3.1.

The next example is provided to illustrate this concept.

Example 3.1:
The density changes in a pipe, due to temperature variation and other reasons, can be approximated as

$$\frac{\rho(x,t)}{\rho_0} = \left(1 - \frac{x}{L}\right)^2 \cos\frac{t}{t_0}.$$

The conduit shown in Figure 3.1 length is L and its area is A. Express the mass flow in and/or out, and the mass in the conduit as a function of time. Write the expression for the mass change in the pipe.

SOLUTION

Here it is very convenient to choose a non-deformable control volume that is inside the conduit dV is chosen as $\pi\, R^2\, dx$. Using equation (3.11), the flow out (or in) is

$$\frac{d}{dt}\int_{c.v.} \rho dV = \frac{d}{dt}\int_{c.v.} \overbrace{\rho_0 \left(1 - \frac{x}{L}\right)^2 \cos\left(\frac{t}{t_0}\right)}^{\rho(t)} \overbrace{\pi\, R^2\, dx}^{dV}$$

The density is not a function of radius, r and angle, θ and they can be taken out the integral as

$$\frac{d}{dt}\int_{c.v.} \rho dV = \pi\, R^2 \frac{d}{dt}\int_{c.v.} \rho_0 \left(1 - \frac{x}{L}\right)^2 \cos\left(\frac{t}{t_0}\right) dx$$

which results in

$$\textbf{Flow Out} = \overbrace{\pi\, R^2}^{A} \frac{d}{dt}\int_0^L \rho_0 \left(1 - \frac{x}{L}\right)^2 \cos\frac{t}{t_0} dx = -\frac{\pi\, R^2\, L\, \rho_0}{3\, t_0} \sin\left(\frac{t}{t_0}\right)$$

The flow out is a function of length, L, and time, t, and is the change of the mass in the control volume.

End Solution

When the control volume is fixed with time, the derivative in equation (3.11) can enter the integral since the boundaries are fixed in time and hence,

Continuity with Fixed b.c.

$$\int_{V_{c.v.}} \frac{d\,\rho}{dt} dV = -\int_{S_{c.v.}} \rho\, U_{rn}\, dA \tag{3.12}$$

Equation (3.12) is simpler than equation (3.11).

In deformable control volume, the left hand side of question (3.11) can be examined further to develop a simpler equation by using the extend Leibniz integral rule for a constant density and result in

$$\frac{d}{dt}\int_{c.v.} \rho\, dV = \overbrace{\int_{c.v.} \overbrace{\frac{d\,\rho}{dt}}^{=0} dV}^{\text{thus, } =0} + \rho \int_{S_{c.v.}} \hat{n} \cdot U_b\, dA = \rho \int_{S_{c.v.}} U_{bn}\, dA \tag{3.13}$$

where U_b is the boundary velocity and U_{bn} is the normal component of the boundary velocity.

Steady State Continuity Deformable

$$\int_{S_{c.v.}} U_{bn}\, dA = \int_{S_{c.v.}} U_{rn}\, dA \tag{3.14}$$

The meaning of the equation (3.14) is the net growth (or decrease) of the Control volume is by net volume flow into it. Example 3.2 illustrates this point.

Example 3.2:
Balloon is attached to a rigid supply in which is supplied by a constant the mass rate, m_i. Calculate the velocity of the balloon boundaries assuming constant density.

SOLUTION

The applicable equation is

$$\int_{c.v.} U_{bn}\, dA = \int_{c.v.} U_{rn}\, dA$$

The entrance is fixed, thus the relative velocity, U_{rn} is

$$U_{rn} = \begin{cases} -U_p & @ \text{ the valve} \\ 0 & \text{every else} \end{cases}$$

Assume equal distribution of the velocity in balloon surface and that the center of the balloon is moving, thus the velocity has the following form

$$U_b = U_x\, \hat{x} + U_{br}\, \hat{r}$$

Where $\hat{x}$ is unit coordinate in x direction and U_x is the velocity of the center and where $\hat{r}$ is unit coordinate in radius from the center of the balloon and U_{br} is the velocity in that direction. The right side of equation (3.14) is the net change due to the boundary is

$$\int_{S_{c.v.}} (U_x\, \hat{x} + U_{br}\, \hat{r}) \cdot \hat{n}\, dA = \overbrace{\int_{S_{c.v.}} (U_x\, \hat{x}) \cdot \hat{n}\, dA}^{\text{center movement}} + \overbrace{\int_{S_{c.v.}} (U_{br}\, \hat{r}) \cdot \hat{n}\, dA}^{\text{net boundary change}}$$

The first integral is zero because it is like movement of solid body and also yield this value mathematically (excises for mathematical oriented student). The second integral (notice $\hat{n} = \hat{r}$) yields

$$\int_{S_{c.v.}} (U_{br}\, \hat{r}) \cdot \hat{n}\, dA = 4\, \pi\, r^2\, U_{br}$$

Substituting into the general equation yields

$$\rho \overbrace{4\, \pi\, r^2}^{A} U_{br} = \rho\, U_p\, A_p = m_i$$

Hence,

$$U_{br} = \frac{m_i}{\rho\, 4\, \pi\, r^2}$$

The center velocity is (also) exactly U_{br}. The total velocity of boundary is

$$U_t = \frac{m_i}{\rho\, 4\, \pi\, r^2}\, (\hat{x} + \hat{r})$$

It can be noticed that the velocity at the opposite to the connection to the rigid pipe which is double of the center velocity.

End Solution

One–Dimensional Control Volume

Additional simplification of the continuity equation is of one dimensional flow. This simplification provides very useful description for many fluid flow phenomena. The main assumption made in this model is that the proprieties in the across section are only function of x coordinate . This assumptions leads

$$\int_{A_2} \rho_2\, U_2\, dA - \int_{A_1} \rho_1\, U_1\, dA = \frac{d}{dt} \int_{V(x)} \rho(x)\, \overbrace{A(x)\, dx}^{dV} \tag{3.15}$$

When the density can be considered constant equation (3.15) is reduced to

$$\int_{A_2} U_2\, dA - \int_{A_1} U_1\, dA = \frac{d}{dt} \int A(x) dx \tag{3.16}$$

For steady state but with variations of the velocity and variation of the density reduces equation (3.15) to become

$$\int_{A_2} \rho_2\, U_2\, dA = \int_{A_1} \rho_1\, U_1\, dA \tag{3.17}$$

For steady state and uniform density and velocity equation (3.17) reduces further to

$$\rho_1\, A_1\, U_1 = \rho_2\, A_2\, U_2 \tag{3.18}$$

For incompressible flow (constant density), continuity equation is at its minimum form of

$$U_1\, A_1 = A_2\, U_2 \tag{3.19}$$

3.3.3 Reynolds Transport Theorem

It can be noticed that the same derivations carried for the density can be carried for other intensive properties such as specific entropy, specific enthalpy. Suppose that f is

intensive property (which can be a scalar or a vector) undergoes change with time. The change of accumulative property will be then

$$\frac{D}{Dt}\int_{sys} f\,\rho dV = \frac{d}{dt}\int_{c.v.} f\,\rho dV + \int_{c.v} f\,\rho\, U_{rn} dA \tag{3.20}$$

This theorem named after Reynolds, Osborne, (1842-1912) which is actually a three dimensional generalization of Leibniz integral rule[1]. To make the previous derivation clearer, the Reynolds Transport Theorem will be reproofed and discussed. The ideas are the similar but extended some what.

Leibniz integral rule[2] is an one dimensional and it is defined as

$$\frac{d}{dy}\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx = \int_{x_1(y)}^{x_2(y)} \frac{\partial f}{\partial y}\,dx + f(x_2,y)\,\frac{dx_2}{dy} - f(x_1,y)\,\frac{dx_1}{dy} \tag{3.21}$$

Initially, a proof will be provided and the physical meaning will be explained. Assume that there is a function that satisfy the following

$$G(x,y) = \int^{x} f\,(\alpha,\,y)\;d\alpha \tag{3.22}$$

Notice that lower boundary of the integral is missing and is only the upper limit of the function is present[3]. For its derivative of equation (3.22) is

$$f(x,y) = \frac{\partial G}{\partial x} \tag{3.23}$$

differentiating (chain rule $d\,uv = u\,dv + v\,du$) by part of left hand side of the Leibniz integral rule (it can be shown which are identical) is

$$\frac{d\,[G(x_2,y) - G(x_1,y)]}{dy} = \overbrace{\frac{\partial G}{\partial x_2}\frac{dx_2}{dy}}^{1} + \overbrace{\frac{\partial G}{\partial y}(x_2,y)}^{2} - \overbrace{\frac{\partial G}{\partial x_1}\frac{dx_1}{dy}}^{3} - \overbrace{\frac{\partial G}{\partial y}(x_1,y)}^{4} \tag{3.24}$$

The terms 2 and 4 in equation (3.24) are actually (the x_2 is treated as a different variable)

$$\frac{\partial G}{\partial y}(x_2,y) - \frac{\partial G}{\partial y}(x_1,y) = \int_{x_1(y)}^{x_2(y)} \frac{\partial\, f(x,y)}{\partial y}\,dx \tag{3.25}$$

The first term (1) in equation (3.24) is

$$\frac{\partial G}{\partial x_2}\frac{dx_2}{dy} = f(x_2,y)\,\frac{dx_2}{dy} \tag{3.26}$$

[1]These papers can be read on-line at http://www.archive.org/details/papersonmechanic01reynrich.

[2]This material is not necessarily but is added her for completeness. This author provides this material just given so no questions will be asked.

[3]There was a suggestion to insert arbitrary constant which will be canceled and will a provide rigorous proof. This is engineering book and thus, the exact mathematical proof is not the concern here. Nevertheless, if there will be a demand for such, it will be provided.

The same can be said for the third term (3). Thus this explanation is a proof the Leibniz rule.

The above "proof" is mathematical in nature and physical explanation is also provided. Suppose that a fluid is flowing in a conduit. The intensive property, f is investigated or the accumulative property, F. The interesting information that commonly needed is the change of the accumulative property, F, with time. The change with time is

$$\frac{DF}{Dt} = \frac{D}{Dt}\int_{sys} \rho\, f\, dV \tag{3.27}$$

For one dimensional situation the change with time is

$$\frac{DF}{Dt} = \frac{D}{Dt}\int_{sys} \rho\, f\, A(x) dx \tag{3.28}$$

If two limiting points (for the one dimensional) are moving with a different coordinate system, the mass will be different and it will not be a system. This limiting condition is the control volume for which some of the mass will leave or enter. Since the change is very short (differential), the flow in (or out) will be the velocity of fluid minus the boundary at x_1, $U_{rn} = U_1 - U_b$. The same can be said for the other side. The accumulative flow of the property in, F, is then

$$F_{in} = \overbrace{f_1\,\rho}^{F_1}\,\overbrace{U_{rn}}^{\frac{dx_1}{dt}} \tag{3.29}$$

The accumulative flow of the property out, F, is then

$$F_{out} = \overbrace{f_2\,\rho}^{F_2}\,\overbrace{U_{rn}}^{\frac{dx_2}{dt}} \tag{3.30}$$

The change with time of the accumulative property, F, between the boundaries is

$$\frac{d}{dt}\int_{c.v.} \rho(x)\, f\, A(x)\, dA \tag{3.31}$$

When put together it brings back the Leibniz integral rule. Since the time variable, t, is arbitrary and it can be replaced by any letter. The above discussion is one of the physical meaning of the Leibniz' rule.

Reynolds Transport theorem is a generalization of the Leibniz rule and thus the same arguments are used. The only difference is that the velocity has three components and only the perpendicular component enters into the calculations.

Reynolds Transport

$$\frac{D}{DT}\int_{sys} f\,\rho dV = \frac{d}{dt}\int_{c.v} f\,\rho\, dV + \int_{S_{c.v.}} f\,\rho\, U_{rn}\, dA \tag{3.32}$$

Example 3.3:
Inflated cylinder is supplied in its center with constant mass flow. Assume that the gas mass is supplied in uniformed way of $m_i\,[kg/m/sec]$. Assume that the cylinder inflated uniformly and pressure inside the cylinder is uniform. The gas inside the cylinder obeys the ideal gas law. The pressure inside the cylinder is linearly proportional to the volume. For simplicity, assume that the process is isothermal. Calculate the cylinder boundaries velocity.

SOLUTION

The applicable equation is

$$\overbrace{\int_{V_{c.v}} \frac{d\rho}{dt}\, dV}^{\text{increase pressure}} + \overbrace{\int_{S_{c.v.}} \rho\, U_b dV}^{\text{boundary velocity}} = \overbrace{\int_{S_{c.v.}} \rho U_{rn}\, dA}^{\text{in or out flow rate}}$$

Every term in the above equation is analyzed but first the equation of state and volume to pressure relationship have to be provided.

$$\rho = \frac{P}{R\,T}$$

and relationship between the volume and pressure is

$$P = f\,\pi\,{R_c}^2$$

Where R_c is the instantaneous cylinder radius. Combining the above two equations results in

$$\rho = \frac{f\,\pi\,{R_c}^2}{R\,T}$$

Where f is a coefficient with the right dimension. It also can be noticed that boundary velocity is related to the radius in the following form

$$U_b = \frac{dR_c}{dt}$$

The first term requires to find the derivative of density with respect to time which is

$$\frac{d\rho}{dt} = \frac{d}{dt}\left(\frac{f\,\pi\,{R_c}^2}{R\,T}\right) = \frac{2\,f\,\pi\,R_c}{R\,T}\overbrace{\frac{dR_c}{dt}}^{U_b}$$

Thus, the first term is

$$\int_{V_{c.v}} \frac{d\rho}{dt}\overbrace{dV}^{2\,\pi\,R_c} = \int_{V_{c.v}} \frac{2\,f\,\pi\,R_c}{R\,T}U_b\overbrace{dV}^{2\,\pi\,R_c\,dR_c} = \frac{4\,f\,\pi^2\,{R_c}^3}{3\,R\,T}U_b$$

The integral can be carried when U_b is independent of the R_c[4] The second term is

$$\int_{S_{c.v.}} \rho\, U_b dA = \overbrace{\frac{f\,\pi\,{R_c}^2}{R\,T}}^{\rho}\, U_b\, \overbrace{2\,\pi R_c}^{A} = \left(\frac{f\,\pi^3\,{R_c}^2}{R\,T}\right) U_b$$

substituting in the governing equation obtained the form of

$$\frac{f\,\pi^2\,{R_c}^3}{R\,T} U_b + \frac{4\,f\,\pi^2\,{R_c}^3}{3\,R\,T}\, U_b = m_i$$

The boundary velocity is then

$$U_b = \frac{m_i}{\dfrac{7\,f\,\pi^2\,{R_c}^3}{3\,R\,T}} G = \frac{3\,m_i\,R\,T}{7\,f\,\pi^2\,{R_c}^3}$$

End Solution

Example 3.4:
A balloon is attached to a rigid supply and is supplied by a constant mass rate, m_i. Assume that gas obeys the ideal gas law. Assume that balloon volume is a linear function of the pressure inside the balloon such as $P = f_v\,V$. Where f_v is a coefficient describing the balloon physical characters. Calculate the velocity of the balloon boundaries under the assumption of isothermal process.

Solution

The question is more complicated than Example 3.4. The ideal gas law is

$$\rho = \frac{P}{R\,T}$$

The relationship between the pressure and volume is

$$P = f_v\,V = \frac{4\,f_v\,\pi\,{R_b}^3}{3}$$

The combining of the ideal gas law with the relationship between the pressure and volume results

$$\rho = \frac{4\,f_v\,\pi\,{R_b}^3}{3\,R\,T}$$

The applicable equation is

$$\int_{V_{c.v}} \frac{d\rho}{dt}\, dV + \int_{S_{c.v.}} \rho\, \left(U_c\,\hat{x} + U_b\hat{r}\right) dA = \int_{S_{c.v.}} \rho U_{rn}\, dA$$

[4]The proof of this idea is based on the chain differentiation similar to Leibniz rule. When the derivative of the second part is $dU_b/dR_c = 0$.

The right hand side of the above equation is

$$\int_{S_{c.v.}} \rho U_{rn}\, dA = m_i$$

The density change is

$$\frac{d\rho}{dt} = \frac{12\, f_v\, \pi\, {R_b}^2}{R\,T} \overbrace{\frac{dR_b}{dt}}^{U_b}$$

The first term is

$$\int_0^{R_b} \overbrace{\frac{12\, f_v\, \pi\, {R_b}^2}{R\,T}}^{\neq f(r)} U_b \overbrace{4\,\pi\, r^2\, dr}^{dV} = \frac{16\, f_v\, \pi^2\, {R_b}^5}{3\,R\,T}\, U_b$$

The second term is

$$\int_A \frac{4\, f_v\, \pi\, {R_b}^3}{3\,R\,T}\, U_b\, dA = \frac{4\, f_v\, \pi\, {R_b}^3}{3\,R\,T}\, U_b \overbrace{4\,\pi\, {R_b}^2}^{A} = \frac{8\, f_v\, \pi^2\, {R_b}^5}{3\,R\,T}\, U_b$$

Subsisting the two equations of the applicable equation results

$$U_b = \frac{1}{8} \frac{m_i\, R\,T}{f_v\, \pi^2\, {R_b}^5}$$

Notice that first term is used to increase the pressure and second the change of the boundary.

End Solution

3.4 Momentum Conservation

In the previous section, the Reynolds Transport Theorem (RTT) was applied to mass conservation. Mass is a scalar (quantity without magnitude). This section deals with momentum conservation which is a vector. The Reynolds Transport Theorem (RTT) can be applicable to any quantity and hence can be apply to vectors. Newton's second law for a single body can apply to multiply body system which further extended to continuous infinitesimal elements. In analysis the Newton's law, it is common to differentiate the external forces into body forces, surface forces. In many problems, the main body force is the gravity which acts on all the system elements.

The surface forces are divided into two categories: one perpendicular to the surface and one in the surface plane. Thus, it can be written as

$$\sum \boldsymbol{F}_s = \int_{c.v.} \boldsymbol{S_n}\, dA + \int_{c.v.} \boldsymbol{\tau}\, dA \tag{3.33}$$

Where the surface "force", $\boldsymbol{S_n}$, is in the surface direction, and $\boldsymbol{\tau}$ are the shear stresses. The surface "force", $\boldsymbol{S_n}$, is made out of two components, one due to viscosity (solid body) and two consequence of the fluid pressure. Assume that the pressure component reasonable to represent $\boldsymbol{S_n}$.

3.4.1 Momentum Governing Equation

Newton's second law $d(m\boldsymbol{U})/dt = \boldsymbol{F}$ requires that the use Reynolds Transport Theorem (RTT) interpretation which is

$$\frac{D}{Dt}\int_{sys}\rho\boldsymbol{U}dV = \frac{t}{dt}\int_{c.v.}\rho\boldsymbol{U}dV + \int_{c.v.}\rho\boldsymbol{U}\boldsymbol{U}_{rn}dA \tag{3.34}$$

Thus, the general form of the momentum equation without the external forces is

Integral Momentum Equation

$$\int_{c.v.}\boldsymbol{g}\,\rho\,dV - \int_{c.v.}\boldsymbol{P}\,dA + \int_{c.v.}\boldsymbol{\tau}\cdot\boldsymbol{dA} = \frac{t}{dt}\int_{c.v.}\rho\boldsymbol{U}dV + \int_{c.v.}\rho\boldsymbol{U}\,\boldsymbol{U_{rn}}\,dV \tag{3.35}$$

With external forces equation (3.35) is transformed to

Integral Momentum Equation & External Forces

$$\sum\boldsymbol{F}_{ext} + \int_{c.v.}\boldsymbol{g}\,\rho\,dV - \int_{c.v.}\boldsymbol{P}\cdot\boldsymbol{dA} + \int_{c.v.}\boldsymbol{\tau}\cdot\boldsymbol{dA} = \frac{t}{dt}\int_{c.v.}\rho\boldsymbol{U}dV + \int_{c.v.}\rho\boldsymbol{U}\,\boldsymbol{U_{rn}}dV \tag{3.36}$$

The external forces, F_{ext}, are the forces resulting from support of the control volume by non–fluid elements. These external forces are commonly associated with pipe, ducts, supporting solid structures, friction (non-fluid), etc.

Equation (3.36) is a vector equation which can be broken into its three components. In Cartesian coordinate, for example in the x coordinate, the components are

$$\sum F_x + \int_{c.v.}\left(\boldsymbol{g}\cdot\hat{i}\right)\rho\,dV\int_{c.v.}\boldsymbol{P}\cos\theta_x\,dA + \int_{c.v.}\boldsymbol{\tau}_x\cdot\boldsymbol{dA} = \frac{t}{dt}\int_{c.v.}\rho\boldsymbol{U}_x\,dV + \int_{c.v.}\rho\boldsymbol{U}_x\cdot\boldsymbol{U}_{rn}dA \tag{3.37}$$

where θ_x is the angle between $\hat{n}$ and $\hat{i}$ or $(\hat{n}\cdot\hat{i})$.

The momentum equation can be simplified for the steady state condition because the unsteady term is zero as

Integral Steady State Momentum Equation

$$\sum\boldsymbol{F}_{ext} + \int_{c.v.}\boldsymbol{g}\,\rho\,dV - \int_{c.v.}\boldsymbol{P}\,dA + \int_{c.v.}\boldsymbol{\tau}\,dA = \int_{c.v.}\rho\boldsymbol{U}U_{rn}dA \tag{3.38}$$

Another important sub category of simplification deals with flow under approximation of the frictionless flow and uniform pressure. This kind of situations arise when

friction (forces) is small compared to kinetic momentum change. Additionally, in these situations, flow is exposed to the atmosphere and thus (almost) uniform pressure surrounding the control volume. In this situation, the mass flow rate in and out are equal. Thus, equation (3.38) is further reduced to

$$\boldsymbol{F} = \int_{out} \rho \boldsymbol{U} \overbrace{(\boldsymbol{U} \cdot \hat{n})}^{U_{rn}} dA - \int_{in} \rho \boldsymbol{U} \overbrace{(\boldsymbol{U} \cdot \hat{n})}^{U_{rn}} dA \tag{3.39}$$

In situations where the velocity is provided and known (remember that the density is constant) the integral can be replaced by

$$\boldsymbol{F} = \dot{m}\overline{\boldsymbol{U}_o} - \dot{m}\overline{\boldsymbol{U}_i} \tag{3.40}$$

The average velocity is related to the velocity profile by the following integral

$$\overline{U}^2 = \frac{1}{A}\int_A [U(r)]^2 \, dA \tag{3.41}$$

Equation (3.41) is applicable to any velocity profile and any geometrical shape.

3.4.2 Conservation Moment of Momentum

The angular momentum can be derived in the same manner as the momentum equation for control volume. The force

$$F = \frac{D}{Dt}\int_{V_{sys}} \rho \boldsymbol{U} dV \tag{3.42}$$

The angular momentum then will be obtained by calculating the change of every element in the system as

$$\mathfrak{M} = \boldsymbol{r} \times \boldsymbol{F} = \frac{D}{Dt}\int_{V_{sys}} \rho \boldsymbol{r} \times \boldsymbol{U} \, dV \tag{3.43}$$

Now the left hand side has to be transformed into the control volume as

$$\mathfrak{M} = \frac{d}{dt}\int_{V_{c.v.}} \rho \, (\boldsymbol{r} \times \boldsymbol{U}) \, dV + \int_{S_{c.v}} \rho \, (\boldsymbol{r} \times \boldsymbol{U}) \boldsymbol{U}_{rn} \, dA \tag{3.44}$$

The angular momentum equation, applying equation (3.44) to uniform and steady state flow with neglected pressure gradient is reduced to

$$\mathfrak{M} = \dot{m} \, (r_2 \times U_2 + r_2 \times U_1) \tag{3.45}$$

Example 3.5:
A large tank has opening with area, A. In front and against the opening there a block with mass of 50[kg]. The friction factor between the block and surface is 0.5. Assume that resistance between the air and the water jet is negligible. Calculated the minimum height of the liquid in the tank in order to start to have the block moving?

SOLUTION

The solution of this kind problem first requires to know at what accuracy this solution is needed. For great accuracy, the effect minor loss or the loss in the tank opening have taken into account. First assuming that a minimum accuracy therefore the information was given on the tank that it large. First, the velocity to move the block can be obtained from the analysis of the block free body diagram (the impinging jet diagram).

The control volume is attached to the block. It is assumed that the two streams in the vertical cancel each other. The jet stream has only one component in the horizontal component. Hence,

$$F = \rho\, A\, {U_{exit}}^2 \qquad \text{(3.V.a)}$$

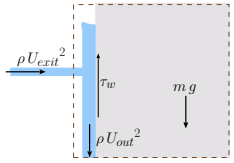


Fig. -3.2. Jet impinging jet surface perpendicular and with the surface.

The minimum force the push the block is

$$\rho\, A\, {U_{exit}}^2 = m\, g\, \mu \Longrightarrow U_{exit} = \sqrt{\frac{m\, g\, \mu}{\rho\, A}} \qquad \text{(3.V.b)}$$

And the velocity as a function of the height is $U = \sqrt{\rho\, g\, h}$ and thus

$$h = \frac{m\, \mu}{\rho^2\, A} \qquad \text{(3.V.c)}$$

It is interesting to point out that the gravity is relevant. That is the gravity has no effect on the velocity (height) required to move the block. However, if the gravity was in the opposite direction, no matter what the height will be the block will not move (neglecting other minor effects). So, the gravity has effect and the effect is the direction, that is the same height will be required on the moon as the earth.

For very tall blocks, the forces that acts on the block in the vertical direction is can be obtained from the analysis of the control volume shown in Figure 3.2. The jet impinged on the surface results in out flow stream going to all the directions in the block surface. Yet, the gravity acts on all these "streams" and eventually the liquid flows downwards. In fact because the gravity the jet impinging in downwards sled direction. At the extreme case, all liquid flows downwards. The balance on the stream downwards (for steady state) is

$$\rho\, \overline{U_{out}}^2 \cong \rho\, V_{liquid}\, g + m\, g \qquad \text{(3.V.d)}$$

Where V_{liquid} is the liquid volume in the control volume (attached to the block). The pressure is canceled because the flow is exposed to air. In cases were $\rho\, V_{liquid}\, g > \rho\, \overline{U_{out}}^2$ the required height is larger. In the opposite cases the height is smaller.

End Solution

3.5 Energy Conservation

This section deals with the energy conservation or the first law of thermodynamics. The fluid, as all phases and materials, obeys this law which creates strange and wonderful phenomena such as a shock and choked flow.

It was shown in Chapter 2 that the energy rate equation (2.10) for a system is

$$\frac{D}{Dt}\left(E_U + m\,\frac{U^2}{2} + m\,g\,z\right) = \dot{Q} - \dot{W} \tag{3.46}$$

Equation (3.46) requires that the time derivative interpretation from a system to a control volume. The energy transfer is carried (mostly[5]) by heat transfer to the system or the control volume. There are three modes of heat transfer, conduction, convection[6] and radiation. In most problems, the radiation is minimal and the discussing will be restricted to convection and conduction. The convection are mostly covered by the terms on the right hand side. The main heat transfer mode on the left hand side is conduction. Conduction for most simple cases is governed by Fourier's Law which is

$$d\dot{q} = k_T \frac{dT}{dn} dA \tag{3.47}$$

Where $d\dot{q}$ is heat transfer to an infinitesimal small area per time and k_T is the heat conduction coefficient. The heat derivative is normalized into area direction. The total heat transfer to the control volume is

$$\dot{Q} = \int_{A_{cv}} k \frac{dT}{dn} dA \tag{3.48}$$

The work done on the system is more complicated to express than the heat transfer. There are two kinds of works that the system does on the surroundings. The first kind work is by the friction or the shear stress and the second by normal force. As in the previous chapter, the surface forces are divided into two categories: one perpendicular to the surface and one with the surface direction. The work done by system on the surroundings (see Figure 3.3) is

$$dw = \overbrace{-\boldsymbol{S}\, d\boldsymbol{A}}^{d\boldsymbol{F}} \cdot d\ell = -\left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \cdot \overbrace{d\boldsymbol{\ell} dA}^{dV} \tag{3.49}$$

The change of the work for an infinitesimal time (excluding the shaft work) is

$$\frac{dw}{dt} = -\left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \cdot \overbrace{\frac{d\boldsymbol{\ell}}{dt}}^{U} dA = -\left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \cdot \boldsymbol{U}\, dA \tag{3.50}$$

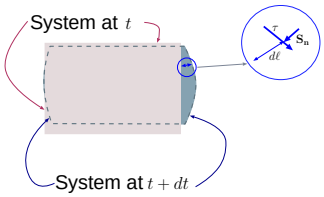


Fig. -3.3. The work on the control volume is done by two different mechanisms, $\mathbf{S_n}$ and τ.

The total work for the system including the shaft work is

$$\dot{W} = -\int_{Ac.v.} \left(\boldsymbol{S_n} + \boldsymbol{\tau}\right) \boldsymbol{U}\, dA - W_{shaft} \tag{3.51}$$

[5]There are other methods such as magnetic fields (like microwave) which are not part of this book.

[6]When dealing with convection, actual mass transfer must occur and thus no convection is possible to a system by the definition of system.

The basic energy equation (3.46) for system is

$$\int_{A_{sys}} k_T \frac{dT}{dn} dA + \int_{A_{sys}} (\boldsymbol{S_n} + \boldsymbol{\tau})\, dV + \dot{W}_{shaft} = \frac{D}{Dt} \int_{V_{sys}} \rho \left(E_U + m\, \frac{U^2}{2} + g\, z \right) dV \tag{3.52}$$

Equation (3.52) does not apply any restrictions on the system. The system can contain solid parts as well several different kinds of fluids. Now Reynolds Transport Theorem can be used to transformed the left hand side of equation (3.52) and thus yields

Energy Equation

$$\int_{A_{cv}} k_T \frac{dT}{dn} dA + \int_{A_{cv}} (\boldsymbol{S_n} + \boldsymbol{\tau})\, dA + \dot{W}_{shaft} = \frac{d}{dt} \int_{V_{cv}} \rho \left(E_u + m\, \frac{U^2}{2} + g\, z \right) dV + \int_{A_{cv}} \left(E_u + m\, \frac{U^2}{2} + g\, z \right) \rho\, U_{rn} dA \tag{3.53}$$

From now on the control volume notation and system will be dropped since all equations deals with the control volume. In the last term in equation (3.53) the velocity appears twice. Note that U is the velocity in the frame of reference while U_{rn} is the velocity relative to the boundary. The normal stress component is replaced by the pressure. The work rate (excluding the shaft work) is

$$\dot{W} \cong \overbrace{\int_S P \hat{n} \cdot \boldsymbol{U} dA}^{\text{flow work}} - \int_S \boldsymbol{\tau} \cdot \boldsymbol{U}\, \hat{n}\, dA \tag{3.54}$$

The first term on the right hand side is referred to in the literature as the flow work and is

$$\int_S P \hat{n} \cdot \boldsymbol{U} dA = \int_S P \overbrace{(U - U_b)\, \hat{n}}^{U_{rn}} dA + \int_S P\, U_{bn} dA \tag{3.55}$$

Equation (3.55) can be further manipulated to become

$$\int_S P \hat{n} \cdot \boldsymbol{U} dA = \overbrace{\int_S \frac{P}{\rho}\, \rho\, U_{rn}\, dA}^{\text{work due to the flow}} + \overbrace{\int_S P U_{bn} dA}^{\text{work due to boundaries movement}} \tag{3.56}$$

The second term is referred to as the shear work and is defined as

$$\dot{W}_{shear} = -\int_S \boldsymbol{\tau} \cdot \boldsymbol{U} dA \tag{3.57}$$

Substituting all these terms into the governing equation yields

$$\begin{aligned} \dot{Q} - \dot{W}_{shear} - \quad \dot{W}_{shaft} = \frac{d}{dt}\int_V \left(E_u + \frac{U^2}{2} + g\,z\right) dV + \\ \int_S \left(E_u + \frac{P}{\rho} + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\,dA + \int_S P U_{rn} dA \end{aligned} \tag{3.58}$$

The new term P/ρ combined with the internal energy, E_u is referred to as the enthalpy, h, which was discussed on page 28. With these definitions equation (3.58) transformed

Simplified Energy Equation

$$\begin{aligned} \dot{Q} - \dot{W}_{shear} + \quad \dot{W}_{shaft} = \frac{d}{dt}\int_V \left(E_u + \frac{U^2}{2} + g\,z\right) \rho\,dV + \\ \int_S \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\,dA + \int_S P U_{bn} dA \end{aligned} \tag{3.59}$$

Equation (3.59) describes the basic energy conservation for the control volume in stationary coordinates.

3.5.1 Approximation of Energy Equation

The energy equation is complicated and several simplifications are commonly used. These simplifications provides reasonable results and key understanding of the physical phenomena and yet with less work.

The steady state situation provides several ways to reduce the complexity. The time derivative term can be eliminated since the time derivative is zero. The acceleration term must be eliminated for the obvious reason. Hence the energy equation is reduced to

Steady State Equation

$$\dot{Q} - \dot{W}_{shear} - \dot{W}_{shaft} = \int_S \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho\,dA + \int_S P U_{bn} dA \tag{3.60}$$

If the flow is uniform or can be estimated as uniform, equation (3.60) is reduced to

Steady State Equation & uniform

$$\begin{aligned} \dot{Q} - \dot{W}_{shear} - \dot{W}_{shaft} = \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho A_{out} - \\ \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho A_{in} + P U_{bn} A_{out} - P U_{bn} A_{in} \end{aligned} \tag{3.61}$$

It can be noticed that last term in equation (3.61) for non-deformable control volume does not vanished. The reason is that while the velocity is constant, the pressure is different. For a stationary fix control volume the energy equation, under this simplification transformed to

$$\dot{Q} - \dot{W}_{shear} - \dot{W}_{shaft} = \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho A_{out} - \left(h + \frac{U^2}{2} + g\,z\right) U_{rn}\,\rho A_{in} \quad (3.62)$$

Dividing equation the mass flow rate provides

Steady State Equation, Fix $\dot{m}$ & uniform

$$\dot{q} - \dot{w}_{shear} - \dot{w}_{shaft} = \left(h + \frac{U^2}{2} + g\,z\right)\Big|_{out} - \left(h + \frac{U^2}{2} + g\,z\right)\Big|_{in} \quad (3.63)$$

Energy Equation in Frictionless Flow and Steady State

In cases were the friction can be neglected using the second law of thermodynamics yields

$$dq_{rev} = dE_u + d\,(P\,v) - v\,dP = dE_u + d\left(\frac{P}{\rho}\right) - v\,dP \quad (3.64)$$

Integrating equation (3.64) and taking time derivative transformed equation (3.64) into Using the RTT to transport equations to control volume results in

$$\dot{Q}_{rev} = \dot{m}\left[(h_{out} - h_{in}) - \left(\int \frac{dP}{\rho}\Big|_{out} - \int \frac{dP}{\rho}\Big|_{in}\right)\right] \quad (3.65)$$

After additional manipulations results in

$$0 = w_{shaft} + \overbrace{\left(\int \frac{dP}{\rho}\Big|_2 - \int \frac{dP}{\rho}\Big|_1\right)}^{\text{change in pressure energy}} + \overbrace{\frac{{U_2}^2 - {U_1}^2}{2}}^{\text{change in kinetic energy}} + \overbrace{g\,(z_2 - z_1)}^{\text{change in potential energy}} \quad (3.66)$$

Equation (3.66) for constant density is

$$0 = w_{shaft} + \frac{P_2 - P_1}{\rho} + \frac{{U_2}^2 - {U_1}^2}{2} + g\,(z_2 - z_1) \quad (3.67)$$

For no shaft work equation (3.67) reduced to

$$0 = \frac{P_2 - P_1}{\rho} + \frac{{U_2}^2 - {U_1}^2}{2} + g\,(z_2 - z_1) \quad (3.68)$$

Example 3.6:

Consider a flow in a long straight pipe. Initially the flow is in a rest. At time, t_0 the

a constant pressure difference is applied on the pipe. Assume that flow is incompressible, and the resistance or energy loss is f. Furthermore assume that this loss is a function of the velocity square. Develop equation to describe the exit velocity as a function of time. State your assumptions.

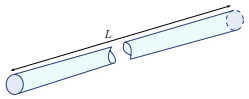


Fig. -3.4. Flow in a long pipe when exposed to a jump in the pressure difference.

SOLUTION

The mass balance on the liquid in the pipe results in

$$0 = \overbrace{\int_V \frac{\partial \rho}{\partial t} dV}^{=0} + \overbrace{\int_A \rho\, U_{bn} dA}^{=0} + \int_A \rho\, U_{rn} dA \Longrightarrow \cancel{\rho A} U_{in} = \cancel{\rho A} U_{exit} \tag{3.VI.a}$$

There is no change in the liquid mass inside pipe and therefore the time derivative is zero (the same mass resides in the pipe at all time). The boundaries do not move and the second term is zero. Thus, the flow in and out are equal because the density is identical. Furthermore, the velocity is identical because the cross area is same.

It can be noticed that for the energy balance on the pipe, the time derivative can enter the integral because the control volume has fixed boundaries. Hence,

$$\begin{aligned} \dot{Q} - \overbrace{\dot{W}_{shear}}^{=0} + \quad \overbrace{\dot{W}_{shaft}}^{=0} = \int_V \frac{d}{dt}\left(E_u + \frac{U^2}{2} + g\, z\right) \rho\, dV + \\ \int_S \left(h + \frac{U^2}{2} + g\, z\right) U_{rn}\, \rho\, dA + \int_S P U_{bn} dA \end{aligned} \tag{3.VI.b}$$

The boundaries shear work vanishes because the same arguments present before (the work, where velocity is zero, is zero. In the locations where the velocity does not vanished, such as in and out, the work is zero because shear stress are perpendicular to the velocity).

There is no shaft work and this term vanishes as well. The first term on the right hand side (with a constant density) is

$$\rho \int_{V_{pipe}} \frac{d}{dt}\left(E_u + \frac{U^2}{2} + \overbrace{g\, z}^{constant}\right) dV = \rho\, U\, \frac{d\, U}{dt} \overbrace{V_{pipe}}^{L\, \pi\, r^2} + \rho \int_{V_{pipe}} \frac{d}{dt}\left(E_u\right)\, dV \tag{3.VI.c}$$

where L is the pipe length, r is the pipe radius, U averaged velocity.

In this analysis, it is assumed that the pipe is perpendicular to the gravity line and thus the gravity is constant. The gravity in the first term and all other terms, related to the pipe, vanish again because the value of z is constant. Also, as can be noticed from equation (3.VI.a), the velocity is identical (in and out). Hence the second term

becomes

$$\int_A \left(h + \cancelto{constant}{\left(\frac{U^2}{2} + g\,z \right)} \right) \rho\, U_{rn} dA = \int_A \overbrace{\left(E_u + \frac{P}{\rho} \right)}^{h} \rho\, U_{rn} dA \tag{3.VI.d}$$

Equation (3.VI.d) can be further simplified (since the area and averaged velocity are constant, additionally notice that $U = U_{rn}$) as

$$\int_A \left(E_u + \frac{P}{\rho} \right) \rho\, U_{rn} dA = \Delta P\, U\, A + \int_A \rho\, E_u\, U_{rn}\, dA \tag{3.VI.e}$$

The third term vanishes because the boundaries velocities are zero and therefore

$$\int_A P\; U_{bn} dA = 0 \tag{3.VI.f}$$

Combining all the terms results in

$$\dot{Q} = \rho\, U\, \frac{d\, U}{dt} \overbrace{V_{pipe}}^{L\,\pi\,r^2} + \rho\, \frac{d}{dt} \int_{V_{pipe}} E_u\, dV + \Delta P\, U\, dA + \int_A \rho\, E_u\, U\, dA \tag{3.VI.g}$$

equation (3.VI.g) can be rearranged as

$$\overbrace{\dot{Q} - \rho \int_{V_{pipe}} \frac{d\,(E_u)}{dt}\, dV - \int_A \rho\, E_u\, U\, dA}^{-K\frac{U^2}{2}} = \rho\, L\, \pi\, r^2\, U\, \frac{d\, U}{dt} + (P_{in} - P_{out})\; U \tag{3.VI.h}$$

The terms on the LHS (left hand side) can be combined. It common to assume (to view) that these terms are representing the energy loss and are a strong function of velocity square[7]. Thus, equation (3.VI.h) can be written as

$$-K\, \frac{U^2}{2} = \rho\, L\, \pi\, r^2\, U\, \frac{d\, U}{dt} + (P_{in} - P_{out})\; U \tag{3.VI.i}$$

Dividing equation (3.VI.i) by $K\,U/2$ transforms equation (3.VI.i) to

$$U + \frac{2\,\rho\, L\, \pi\, r^2}{K} \frac{d\, U}{dt} = \frac{2\,(P_{in} - P_{out})}{K} \tag{3.VI.j}$$

Equation (3.VI.j) is a first order differential equation. The solution this equation is described in the appendix and which is

$$U = \mathrm{e}^{-\left(\frac{t\, K}{2\,\pi\, r^2\, \rho\, L} \right)} \left(\frac{2\,(P_{in} - P_{out})\, \mathrm{e}^{\left(\frac{t\, K}{2\,\pi\, r^2\, \rho\, L} \right)}}{K} + c \right) \mathrm{e}^{\left(\frac{2\,\pi\, r^2\, \rho\, t\, L}{K} \right)} \tag{3.VI.k}$$

[7] The shear work inside the liquid refers to molecular work (one molecule work on the other molecule). This shear work can be viewed also as one control volume work on the adjoined control volume.

Applying the initial condition, $U(t=0)=0$ results in

$$U = \frac{2\,(P_{in}-P_{out})}{K}\left(1-\mathrm{e}^{-\left(\frac{t\,K}{2\,\pi\,r^2\,\rho\,L}\right)}\right) \tag{3.VI.l}$$

The solution is an exponentially approaching the steady state solution. In steady state the flow equation (3.VI.j) reduced to a simple linear equation. The solution of the linear equation and the steady state solution of the differential equation are the same.

$$U = \frac{2\,(P_{in}-P_{out})}{K} \tag{3.VI.m}$$

Another note, in reality the resistance, K, is not constant but rather a strong function of velocity (and other parameters such as temperature[8], velocity range, velocity regime and etc.). This function will be discussed in a greater extent later on. Additionally, it should be noted that if momentum balance was used a similar solution (but not the same) was obtained.

End Solution

3.6 Limitations of Integral Approach

The integral method has limit accuracy and some techniques suggested in "Basic of Fluid Mechanics" by Bar–Meir and others are available to enhance the calculations quality. However, even with these enhancements simply cannot tackle some of the problems. The improvements to the integral methods are the corrections to the estimates of the energy or other quantities in the conservation equations. The accuracy issues that integral methods intrinsically suffers from no ability to exact flow field and thus lost the accuracy. The integral method does not handle the problems such as the free surface with reasonable accuracy. In addition, the dissipation can be ignored. In some cases that dissipation play major role which the integral methods ignores. The discussion on the limitations was not provided to discard usage of this method but rather to provide a guidance of use with caution. The integral method is a powerful and yet simple method but has has to be used with the limitations of the method in mind.

3.7 Differential Analysis

The integral analysis has a limited accuracy, which leads to a different approach of differential analysis. The differential analysis allows the flow field investigation in greater detail. In differential analysis, the emphasis is on infinitesimal scale and thus the analysis provides better accuracy as complementary analysis to the integral analysis. This analysis leads to partial differential equations which are referred to as the Navier-Stokes equations. Navier-Stokes equations are non–linear and there are more than one possible

[8] Via the viscosity effects.

solution in many cases (if not most cases) e.g. the solution is not unique. However even for the "regular" solution the mathematics is very complex. Even for simple situations, there are cases when complying with the boundary conditions leads to a discontinuity (shock) or pushes the boundary condition(s) further downstream (choked flow). These issues are discussed later.

3.7.1 Mass Conservation

Fluid flows into and from a three dimensional infinitesimal control volume depicted in Figure 3.5. The mass conservation for this infinitesimal small system is zero thus

$$\frac{D}{Dt}\int_V \rho dV = 0 \qquad (3.69)$$

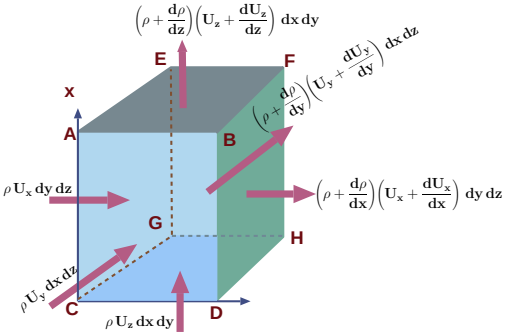


Fig. -3.5. The mass balance on the infinitesimal control volume.

However for a control volume using Reynolds Transport Theorem (RTT), the following can be written

$$\frac{D}{Dt}\int_V \rho dV = \frac{d}{dt}\int_V \rho dV + \int_A U_{rn}\,\rho\, dA = 0 \qquad (3.70)$$

Using the regular interpolation[9] results in

Continuity in Cartesian Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho U_x}{\partial x} + \frac{\partial \rho U_y}{\partial y} + \frac{\partial \rho U_z}{\partial z} = 0 \qquad (3.71)$$

In cylindrical coordinates equation (3.71) is written as

Continuity in Cylindrical Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\,\rho\, U_r)}{\partial r} + \frac{1}{r}\frac{\partial \rho U_\theta}{\partial \theta} + \frac{\partial \rho U_z}{\partial z} = 0 \qquad (3.72)$$

For the spherical coordinates, the continuity equation becomes

Continuity in Spherical Coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial \left(r^2\,\rho\, U_r\right)}{\partial r} + \frac{1}{r\,\sin\theta}\frac{\partial \left(\rho\, U_\theta\,\sin\theta\right)}{\partial \theta} + \frac{1}{r\,\sin\theta}\frac{\partial \rho U_\phi}{\partial z} = 0 \qquad (3.73)$$

[9] See for more details in "Basic of Fluid Mechanics, Bar-Meir, Potto Project, www.potto.org

The continuity equations (3.71), (3.72) and (3.73) can be expressed in a vector form as

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{U}) = 0 \tag{3.74}$$

The use of these equations is normally combined with other equations (momentum and or energy equations). There are very few cases where this equation is used on its own merit.

3.7.2 Momentum Equations or N–S equations

Newton second law first described as an integral equation. Now this integral equation applied to infinitesimal control volume yield differential equation in the x–coordinate of

$$\rho \frac{DU_y}{Dt} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + \rho f_{G_y} \tag{3.75}$$

There are two more equations for the other two coordinates. This equation in vector is

Momentum Equation

$$\rho \frac{D\boldsymbol{U}}{Dt} = \nabla \cdot \boldsymbol{\tau}^{(i)} + \rho \boldsymbol{f_G} \tag{3.76}$$

where here

$$\boldsymbol{\tau}^{(i)} = \tau_{ix}\widehat{i} + \tau_{iy}\widehat{j} + \tau_{iz}\widehat{k}$$

is part of the shear stress tensor and i can be any of the x, y, or z.

Or in index (Einstein) notation as

$$\rho \frac{DU_i}{Dt} = \frac{\partial \tau_{ji}}{\partial x_i} + \rho f_{Gi} \tag{3.77}$$

Equations (3.76) requires that the stress tensor be defined in term of the velocity/deformation. The relationship between the stress tensor and deformation depends on the materials. As engineers do in general, the simplest model is assumed which referred as the solid continuum model. In this model the relationship between the (shear) stresses and rate of strains are assumed to be linear. In solid material, the shear stress yields a fix amount of deformation. In contrast, when applying the shear stress in fluids, the result is a continuous deformation. Furthermore, reduction of the shear stress does not return the material to its original state as in solids. The similarity to solids the increase shear stress in fluids yields larger deformations (larger rate of deformations). Thus this "solid" model is a linear relationship with three main assumptions:

a. There is no preference in the orientation (also call isentropic fluid),

b. there is no left over stresses (In other words when the "no shear stress" situation exist the rate of deformation or strain is zero), and

c. a linear relationship exist between the shear stress and the rate of shear strain.

It was shown[10] that

$$\tau_{ij} = \mu \frac{D\gamma_{ij}}{Dt} = \mu \left(\frac{dU_j}{di} + \frac{dU_i}{dj} \right) \tag{3.78}$$

where $i \neq j$ and $i = x$ or y or z.

After considerable derivations it can be shown that the relationship between the shear stress and the velocity is

$$\tau_{xx} = -P_m + 2\,\mu \frac{\partial U_x}{\partial x} + \frac{2}{3}\mu\, \nabla \cdot \boldsymbol{U} \tag{3.79}$$

where P_m is the mechanical pressure and is defined as

Mechanical Pressure

$$P_m = -\frac{\tau_{xx} + \tau_{yy} + \tau_{zz}}{3} \tag{3.80}$$

Commonality engineers like to combined the two difference expressions into one as

$$\tau_{xy} = -\left(P_m + \frac{2}{3}\mu \nabla \cdot \boldsymbol{U} \right) \overbrace{\delta_{xy}}^{=0} + \mu \left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right) \tag{3.81}$$

or

$$\tau_{xx} = -\left(P_m + \frac{2}{3}\mu \nabla \cdot \boldsymbol{U} \right) \overbrace{\delta_{xy}}^{=1} + \mu \left(\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} \right) \tag{3.82}$$

where δ_{ij} is the Kronecker delta what is $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ otherwise. Or index notation

$$\tau_{ij} = -\left(P_m + \frac{2}{3}\mu \nabla \cdot \boldsymbol{U} \right) \delta_{ij} + \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \tag{3.83}$$

This expression suggests a new definition of the thermodynamical pressure is

Thermodynamic Pressure

$$P = P_m + \frac{2}{3}\mu \nabla \cdot \boldsymbol{U} \tag{3.84}$$

[10] "Basic of Fluid Mechanics", Bar-Meir

Thus, the momentum equation can be written as

$$\rho\left(\frac{DU_x}{Dt}\right) = -\frac{\partial\left(P + \left(\frac{2}{3}\mu - \lambda\right)\nabla\cdot\boldsymbol{U}\right)}{\partial x} + \mu\left(\frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} + \frac{\partial^2 U_x}{\partial z^2}\right) \quad + \boldsymbol{f}_{Bx} \tag{3.85}$$

or in a vector form as

N-S in stationary Coordinates

$$\rho\frac{D\boldsymbol{U}}{Dt} = -\nabla P + \left(\frac{1}{3}\mu + \lambda\right)\nabla\left(\nabla\cdot\boldsymbol{U}\right) + \mu\nabla^2\boldsymbol{U} + \boldsymbol{f}_B \tag{3.86}$$

For in index form as

$$\rho\frac{D\,U_i}{Dt} = -\frac{\partial}{\partial x_i}\left(P + \left(\frac{2}{3}\mu - \lambda\right)\nabla\cdot\boldsymbol{U}\right) + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right) + \boldsymbol{f}_{Bi} \tag{3.87}$$

3.7.3 Boundary Conditions and Driving Forces

Boundary Conditions Categories

The governing equations discussed earlier requires some boundary conditions and initial conditions. These conditions described the physical situations which are believed or should exist or approximated. These conditions can be categorized by the velocity, pressure, or in more general terms as the shear stress conditions. A common velocity condition is that the liquid has the same value as the solid interface velocity which is known as the "no slip" condition. The solid surface is rough thus the liquid participles (or molecules) are slowed to be at the solid surface velocity. This boundary condition was experimentally observed under many conditions yet it is not universally true. The slip condition (as oppose to "no slip" condition) exist in situations where the scale is very small and the velocity is relatively very small. The slip can be neglected in the large scale while the slip cannot be neglected in the small scale.

As oppose to a given velocity at particular point, A boundary condition can be given as requirement on the acceleration (velocity) at unknown location. This condition is called the kinematic boundary condition and associated with liquid and will not be discussed here.

The second condition that commonality prescribed at the interface is the static pressure at a specific location. The static pressure is measured perpendicular to the flow direction. The last condition is similar to the pressure condition of prescribed shear stress or a relationship to it. In this category include the boundary conditions with issues of surface tension.

The body forces, in general and gravity in a particular, are the condition that given on the flow beside the velocity, shear stress (including the surface tension) and the pressure. The gravity is a common body force which is considered in many fluid mechanics problems. The gravity can be considered as a constant force in most cases. Another typical driving force is the shear stress.

CHAPTER 4

Speed of Sound

4.1 Motivation

In traditional compressible flow classes there is very little discussion about the speed of sound outside the ideal gas. The author thinks that this approach has many shortcomings. In a recent consultation an engineer[1] design a industrial system that contains converting diverging nozzle with filter to remove small particles from air. The engineer was well aware of the calculation of the nozzle. Thus, the engineer was able to predict that was a chocking point. Yet, the engineer was not ware of the effect of particles on the speed of sound. Hence, the actual flow rate was only half of his prediction. As it will shown in this chapter, the particles can, in some situations, reduces the speed of sound by almost as half. With the "new" knowledge from the consultation the calculations were within the range of acceptable results.

The above situation is not unique in the industry. It should be expected that engineers know how to manage this situation of non pure substances (like clean air). The fact that the engineer knows about the chocking is great but it is not enough for today's sophisticated industry[2]. In this chapter an introductory discussion is given about different situations which can appear the industry in regards to speed of sound.

4.2 Introduction

[1] Aerospace engineer, alumni of University of Minnesota, Aerospace Department.

[2] Pardon, but a joke is must in this situation. A cat is pursuing a mouse and the mouse escape and hide in the hole. Suddenly, the mouse hear a barking dog and a cat yelling. The mouse go out to investigate, and cat caught the mouse. The mouse asked the cat I thought I heard a dog. The cat reply, yes you did. My teacher was right, one language is not enough today.

The people had recognized for several hundred years that sound is a variation of pressure. The ears sense the variations by frequency and magnitude which are transferred to the brain which translates to voice. Thus, it raises the question: what is the speed of the small disturbance travel in a "quiet" medium. This velocity is referred to as the speed of sound.

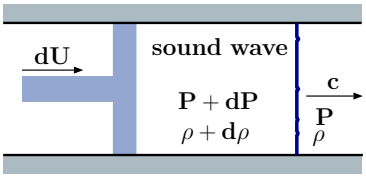


Fig. -4.1. A very slow moving piston in a still gas.

To answer this question consider a piston moving from the left to the right at a relatively small velocity (see Figure 4.1). The information that the piston is moving passes thorough a single "pressure pulse." It is assumed that if the velocity of the piston is infinitesimally small, the pulse will be infinitesimally small. Thus, the pressure and density can be assumed to be continuous.

In the control volume it is convenient to look at a control volume which is attached to a pressure pulse. Applying the mass balance yields

$$\rho c = (\rho + d\rho)(c - dU) \tag{4.1}$$

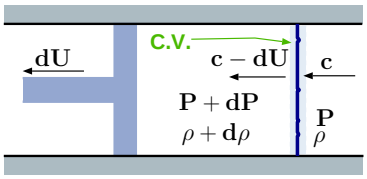


Fig. -4.2. Stationary sound wave and gas moves relative to the pulse.

or when the higher term $dU\,d\rho$ is neglected yields

$$\rho dU = cd\rho \Longrightarrow dU = \frac{cd\rho}{\rho} \tag{4.2}$$

From the energy equation (Bernoulli's equation), assuming isentropic flow and neglecting the gravity results

$$\frac{(c - dU)^2 - c^2}{2} + \frac{dP}{\rho} = 0 \tag{4.3}$$

neglecting second term (dU^2) yield

$$-cdU + \frac{dP}{\rho} = 0 \tag{4.4}$$

Substituting the expression for dU from equation (4.2) into equation (4.4) yields

Sound Speed

$$c^2\left(\frac{d\rho}{\rho}\right) = \frac{dP}{\rho} \Longrightarrow c^2 = \frac{dP}{d\rho} \tag{4.5}$$

An expression is needed to represent the right hand side of equation (4.5). For an ideal gas, P is a function of two independent variables. Here, it is considered that $P = P(\rho, s)$

where s is the entropy. The full differential of the pressure can be expressed as follows:

$$dP = \left.\frac{\partial P}{\partial \rho}\right|_s d\rho + \left.\frac{\partial P}{\partial s}\right|_\rho ds \tag{4.6}$$

In the derivations for the speed of sound it was assumed that the flow is isentropic, therefore it can be written

$$\frac{dP}{d\rho} = \left.\frac{\partial P}{\partial \rho}\right|_s \tag{4.7}$$

Note that the equation (4.5) can be obtained by utilizing the momentum equation instead of the energy equation.

Example 4.1:
Demonstrate that equation (4.5) *can be derived from the momentum equation.*

SOLUTION

The momentum equation written for the control volume shown in Figure (4.2) is

$$\overbrace{(P + dP) - P}^{\sum F} = \overbrace{(\rho + d\rho)(c - dU)^2 - \rho\, c^2}^{\int_{cs} U\,(\rho\, U\, dA)} \tag{4.8}$$

Neglecting all the relative small terms results in

$$dP = (\rho + d\rho)\left(c^2 - \cancelto{\sim 0}{2c dU} + \cancelto{\sim 0}{dU^2}\right) - \rho c^2 \tag{4.9}$$

And finally it becomes

$$dP = c^2\, d\rho \tag{4.10}$$

This yields the same equation as (4.5).

End Solution

4.3 Speed of Sound in Ideal and Perfect Gases

The speed of sound can be obtained easily for the equation of state for an ideal gas (also perfect gas as a sub set) because of a simple mathematical expression. The pressure for an ideal gas can be expressed as a simple function of density, ρ, and a function "molecular structure" or ratio of specific heats, k namely

$$P = constant \times \rho^k \tag{4.11}$$

and hence

$$c = \sqrt{\frac{dP}{d\rho}} = k \times constant \times \rho^{k-1} = k \times \frac{\overbrace{constant \times \rho^{k}}^{P}}{\rho}$$
$$= k \times \frac{P}{\rho} \tag{4.12}$$

Remember that P/ρ is defined for an ideal gas as RT, and equation (4.12) can be written as

Gas Speed Sound

$$c = \sqrt{k\,R\,T} \tag{4.13}$$

Example 4.2:
Calculate the speed of sound in water vapor at $20[bar]$ *and* $350^{\circ}\mathrm{C}$*, (a) utilizes the steam table (b) assuming ideal gas.*

SOLUTION

The solution can be estimated by using the data from steam table[3]

$$c \sim \sqrt{\frac{\Delta P}{\Delta \rho}}_{s=constant} \tag{4.14}$$

At $20[bar]$ and $350^{\circ}\mathrm{C}$: s = 6.9563 $\left[\frac{kJ}{K\ kg}\right]$ ρ = 6.61376 $\left[\frac{kg}{m^3}\right]$
At $18[bar]$ and $350^{\circ}\mathrm{C}$: s = 7.0100 $\left[\frac{kJ}{K\ kg}\right]$ ρ = 6.46956 $\left[\frac{kg}{m^3}\right]$
At $18[bar]$ and $300^{\circ}\mathrm{C}$: s = 6.8226 $\left[\frac{kJ}{K\ kg}\right]$ ρ = 7.13216 $\left[\frac{kg}{m^3}\right]$

After interpretation of the temperature:
At $18[bar]$ and $335.7^{\circ}\mathrm{C}$: s $\sim$ 6.9563 $\left[\frac{kJ}{K\ kg}\right]$ $\rho \sim$ 6.94199 $\left[\frac{kg}{m^3}\right]$
and substituting into the equation yields

$$c = \sqrt{\frac{200000}{0.32823}} = 780.5\left[\frac{m}{sec}\right] \tag{4.15}$$

for ideal gas assumption (data taken from Van Wylen and Sontag, Classical Thermodynamics, table A 8.)

$$c = \sqrt{k\,R\,T} \sim \sqrt{1.327 \times 461 \times (350 + 273)} \sim 771.5\left[\frac{m}{sec}\right]$$

[3]This data is taken from Van Wylen and Sontag "Fundamentals of Classical Thermodynamics" 2nd edition

Note that a better approximation can be done with a steam table, and it

End Solution

Example 4.3:
The temperature in the atmosphere can be assumed to be a linear function of the height for some distances. What is the time it take for sound to travel from point "A" to point "B" under this assumption.?

SOLUTION

The temperature is denoted at "A" as T_A and temperature in "B" is T_B. The distance between "A" and "B" is denoted as h.

$$T\ (x) = T_A + \frac{x}{h}\ (T_B - T_A) = T_A + \frac{x}{h}\left(\frac{T_B}{T_A} - 1\right) T_A \tag{4.16}$$

Where x is the variable distance. It can be noticed[4] that the controlling dimension is the ratio of the edge temperatures. It can be further noticed that the square root of this ratio is affecting parameter and thus this ratio can be defined as

$$\omega = \sqrt{\frac{T_B}{T_A}} \tag{4.17}$$

Using the definition (4.17) in equation (4.16) results in

$$T\ (x) = T_A \left(1 + \frac{\omega^2 - 1}{h} x\right) \tag{4.18}$$

It should be noted that velocity is provided as a function of the distance and not the time (another reverse problem). For an infinitesimal time $d\,\tau$ is equal to

$$d\,\tau = \frac{dx}{\sqrt{k\,R\,T(x)}} = \frac{dx}{\sqrt{k\,R\,T_A \left(1 + \frac{\omega^2 - 1}{h} x\right)}}$$

or the integration the about equation as

$$\int_0^t d\,\tau = \int_0^h \frac{dx}{\sqrt{k\,R\,T_A \left(1 + \frac{\omega^2 - 1}{h} x\right)}}$$

The result of the integration of the above equation yields

$$t_{corrected} = \frac{2\,h}{(w+1)\,\sqrt{k\,R\,T_A}} \tag{4.19}$$

[4]This suggestion was proposed by Heru Reksoprodjo from Helsinki University of Technology, Finland.

For assumption of constant temperature the time is

$$t = \frac{h}{\sqrt{k\,R\,\bar{T_A}}} \tag{4.20}$$

Hence the correction factor

$$\frac{t_{corrected}}{t} = \frac{2}{(w+1)} \tag{4.21}$$

This correction factor approaches one when $T_B \longrightarrow T_A$ because $\omega \longrightarrow 1$.

Another possible question[5] to find the temperature, T_C, where The "standard" equation can be used.

$$\frac{h}{\sqrt{k\,R\,T_C}} = \frac{2\,h}{(w+1)\,\sqrt{k\,R\,T_A}}$$

The above equation leads to

$$T_C = \frac{T_A + T_B + 2\,\sqrt{T_A\,T_B}}{4}$$

The explanation to the last equation is left as exercise to the reader.

End Solution

4.4 Speed of Sound in Real Gases

The ideal gas model can be improved by introducing the compressibility factor. The compressibility factor represents the deviation from the ideal gas.

Thus, a real gas equation can be expressed in many cases (see also page 32) as

$$P = z\,\rho\,R\,T \tag{4.22}$$

The speed of sound of any gas is provided by equation (4.7). To obtain the expression for a gas that obeys the law expressed by (4.22) some mathematical expressions are needed. Recalling from thermodynamics, the Gibbs function (4.23) is used to obtain

$$Tds = dh - \frac{dP}{\rho} \tag{4.23}$$

The definition of pressure specific heat for a pure substance is

$$C_p = \left(\frac{\partial h}{\partial T}\right)_P = T\left(\frac{\partial s}{\partial T}\right)_P \tag{4.24}$$

The definition of volumetric specific heat for a pure substance is

$$C_v = \left(\frac{\partial u}{\partial T}\right)_\rho = T\left(\frac{\partial s}{\partial T}\right)_\rho \tag{4.25}$$

[5] Indirectly was suggested by Heru Reksoprodjo.

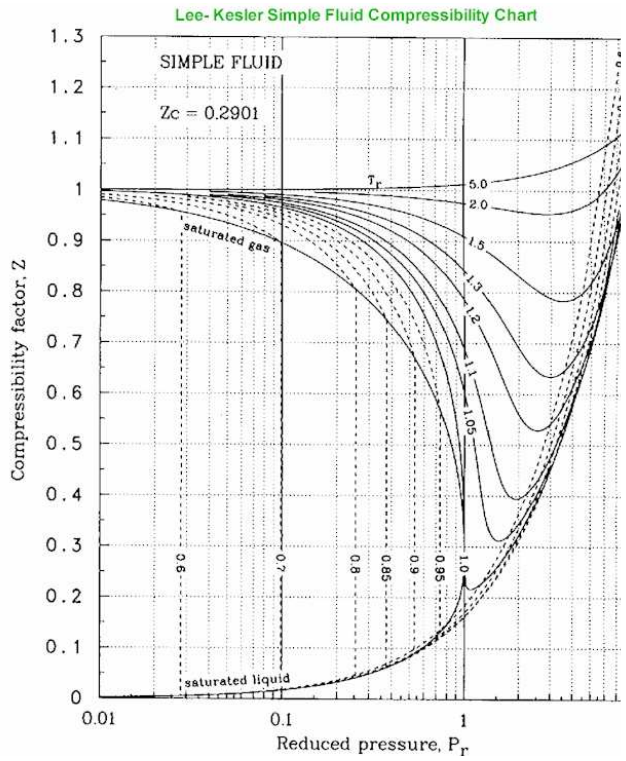


Fig. -4.3. The compressibility chart.

From thermodynamics, it can be shown [6]

$$dh = C_p dT + \left[v - T \left(\frac{\partial v}{\partial T} \right)_P \right] \tag{4.26}$$

The specific volumetric is the inverse of the density as $v = z\,RT/P$ and thus

$$\left(\frac{\partial v}{\partial T} \right)_P = \left(\frac{\partial \left(\frac{z\,RT}{P} \right)}{\partial T} \right)_P = \frac{RT}{P} \left(\frac{\partial z}{\partial T} \right)_P + \frac{z\,R}{P} \cancelto{1}{\left(\frac{\partial T}{\partial T} \right)_P} \tag{4.27}$$

[6] See Van Wylen p. 372 SI version, perhaps to insert the discussion here.

Substituting the equation (4.27) into equation (4.26) results

$$dh = C_p\, dT + \left[v - T \left(\overbrace{\frac{R\,T}{P}}^{\frac{v}{z}} \left(\frac{\partial z}{\partial T} \right)_P + \overbrace{\frac{z\,R}{P}}^{\frac{v}{T}} \right) \right] dP \tag{4.28}$$

Simplifying equation (4.28) to became

$$dh = C_p\, dT - \left[\frac{T\,v}{z} \left(\frac{\partial z}{\partial T} \right)_P \right] dP = C_p dT - \frac{T}{z} \left(\frac{\partial z}{\partial T} \right)_P \frac{dP}{\rho} \tag{4.29}$$

Utilizing Gibbs equation (4.23)

$$\begin{aligned} Tds =& \overbrace{C_p dT - \frac{T}{z} \left(\frac{\partial z}{\partial T} \right)_P \frac{dP}{\rho}}^{dh} - \frac{dP}{\rho} = C_p dT - \frac{dP}{\rho} \left[\frac{T}{z} \left(\frac{\partial z}{\partial T} \right)_P + 1 \right] \\ =& C_p\, dT - \frac{dP}{P} \overbrace{\frac{P}{\rho}}^{z\,R\,T} \left[\frac{T}{z} \left(\frac{\partial z}{\partial T} \right)_P + 1 \right] \end{aligned} \tag{4.30}$$

Letting $ds = 0$ for isentropic process results in

$$\frac{dT}{T} = \frac{dP}{P} \frac{R}{C_p} \left[z + T \left(\frac{\partial z}{\partial T} \right)_P \right] \tag{4.31}$$

Equation (4.31) can be integrated by parts. However, it is more convenient to express dT/T in terms of C_v and $d\rho/\rho$ as follows

$$\frac{dT}{T} = \frac{d\rho}{\rho} \frac{R}{C_v} \left[z + T \left(\frac{\partial z}{\partial T} \right)_\rho \right] \tag{4.32}$$

Equating the right hand side of equations (4.31) and (4.32) results in

$$\frac{d\rho}{\rho} \frac{R}{C_v} \left[z + T \left(\frac{\partial z}{\partial T} \right)_\rho \right] = \frac{dP}{P} \frac{R}{C_p} \left[z + T \left(\frac{\partial z}{\partial T} \right)_P \right] \tag{4.33}$$

Rearranging equation (4.33) yields

$$\frac{d\rho}{\rho} = \frac{dP}{P} \frac{C_v}{C_p} \left[\frac{z + T \left(\frac{\partial z}{\partial T} \right)_P}{z + T \left(\frac{\partial z}{\partial T} \right)_\rho} \right] \tag{4.34}$$

If the terms in the braces are constant in the range under interest in this study, equation (4.34) can be integrated. For short hand writing convenience, n is defined as

$$n = \overbrace{\frac{C_p}{C_v}}^{k} \left(\frac{z + T\left(\frac{\partial z}{\partial T}\right)_\rho}{z + T\left(\frac{\partial z}{\partial T}\right)_P} \right) \tag{4.35}$$

Note that n approaches k when $z \to 1$ and when z is constant. The integration of equation (4.34) yields

$$\left(\frac{\rho_1}{\rho_2}\right)^n = \frac{P_1}{P_2} \tag{4.36}$$

Equation (4.36) is similar to equation (4.11). What is different in these derivations is that a relationship between coefficient n and k was established. This relationship (4.36) isn't new, and in–fact any thermodynamics book shows this relationship. But the definition of n in equation (4.35) provides a tool to estimate n. Now, the speed of sound for a real gas can be obtained in the same manner as for an ideal gas.

Real Gas Sound Speed

$$\frac{dP}{d\rho} = \sqrt{n\, z\, R T} \tag{4.37}$$

Example 4.4:
Calculate the speed of sound of air at $30^\circ\mathrm{C}$ *and atmospheric pressure* $\sim 1[bar]$. *The specific heat for air is* $k = 1.407$, $n = 1.403$, *and* $z = 0.995$.

Make the calculation based on the ideal gas model and compare these calculations to real gas model (compressibility factor). Assume that $R = 287[j/kg/K]$.

SOLUTION

According to the ideal gas model the speed of sound should be

$$c = \sqrt{k\, R\, T} = \sqrt{1.407 \times 287 \times 300} \sim 348.1[m/sec]$$

For the real gas first coefficient $n = 1.403$ has

$$c = \sqrt{z\, n\, R\, T} = \sqrt{1.403 \times 0.995 \times 287 \times 300} = 346.7[m/sec]$$

End Solution

The correction factor for air under normal conditions (atmospheric conditions or even increased pressure) is minimal on the speed of sound. However, a change in temperature can have a dramatical change in the speed of sound. For example, at relative moderate pressure but low temperature common in atmosphere, the compressibility factor, $z = 0.3$ and $n \sim 1$ which means that speed of sound is only $\sqrt{\frac{0.3}{1.4}}$ about factor of (0.5) to calculated by ideal gas model.

4.5 Speed of Sound in Almost Incompressible Liquid

Even liquid *normally* is assumed to be incompressible in reality has a small and important compressible aspect. The ratio of the change in the fractional volume to pressure or compression is referred to as the bulk modulus of the material. For example, the average bulk modulus for water is $2.2 \times 10^9\ N/m^2$. At a depth of about 4,000 meters, the pressure is about $4 \times 10^7\ N/m^2$. The fractional volume change is only about 1.8% even under this pressure nevertheless it is a change.

The compressibility of the substance is the reciprocal of the bulk modulus. The amount of compression of almost all liquids is seen to be very small as given in Table (4.5). The mathematical definition of bulk modulus as following

$$B = \rho \frac{dP}{d\rho} \tag{4.38}$$

In physical terms can be written as

Liquid/Solid Sound Speed

$$c = \sqrt{\frac{elastic\ \ property}{inertial\ \ property}} = \sqrt{\frac{B}{\rho}} \tag{4.39}$$

For example for water

$$c = \sqrt{\frac{2.2 \times 10^9 N/m^2}{1000 kg/m^3}} = 1493 m/s$$

This agrees well with the measured speed of sound in water, 1482 m/s at 20°C. Many researchers have looked at this velocity, and for purposes of comparison it is given in Table (4.5)

The effect of impurity and temperature is relatively large, as can be observed from the equation (4.40). For example, with an increase of 34 degrees from 0°C there is an increase in the velocity from about 1430 m/sec to about 1546 [m/sec]. According to Wilson[7], the speed of sound in sea water depends on temperature, salinity, and hydrostatic pressure.

Wilson's empirical formula appears as follows:

$$c(S, T, P) = c_0 + c_T + c_S + c_P + c_{STP}, \tag{4.40}$$

where $c_0 = 1449.14[m/sec]$ is about clean/pure water, c_T is a function temperature, and c_S is a function salinity, c_P is a function pressure, and c_{STP} is a correction factor between coupling of the different parameters.

[7] J. Acoust. Soc. Amer., 1960, vol.32, N 10, p. 1357. Wilson's formula is accepted by the National Oceanographic Data Center (NODC) USA for computer processing of hydrological information.

Remark	reference	Value [m/sec]
Fresh Water (20 °C)	Cutnell, John D. & Kenneth W. Johnson. Physics. New York: Wiley, 1997: 468.	1492
Distilled Water at (25 °C)	The World Book Encyclopedia. Chicago: World Book, 1999. 601	1496
Water distilled	Handbook of Chemistry and Physics. Ohio: Chemical Rubber Co., 1967-1968:E37	1494

Table -4.1. Water speed of sound from different sources

material	reference	Value [m/sec]
Glycerol		1904
Sea water	25°C	1533
Mercury		1450
Kerosene		1324
Methyl alcohol		1143
Carbon tetrachloride		926

Table -4.2. Liquids speed of sound, after Aldred, John, Manual of Sound Recording, London: Fountain Press, 1972

4.5.1 Sound in Variable Compressible Liquids

The derivations for the calculations of the speed of sound were first presented in Potto Project book *Basics of Fluid Mechanics*, 2009, Potto Project.). It was found that Pushka equation describes the density in a deep sea. The density varies due to the compression created by the liquid above. Pushka equation is written as as for non-geological system in dimensionless form as

Pushka Equation

$$\frac{\rho}{\rho_0} = \sqrt{\frac{1}{\frac{2\,g\,\rho_0\,x}{B_T} + 1}} \tag{4.41}$$

Equation (4.41) models only constant bulk modulus situations. The local speed of sound is described by equation (4.39).

The sound speed changes since the density changes with the depth of liquid. The way to calculate speed of propagation by look at infinitesimal distance. The equation for the sound speed is taken here as correct for very local point. However, the density is different for every point since the density varies and the density is a function of the depth. The speed of sound at any depth point, x, is

$$c = \sqrt{\frac{B_T}{\rho_0 \sqrt{\frac{1}{\frac{2\,g\,\rho_0\,x}{B_T}+1}}}} = \sqrt{\frac{B_T}{\rho_0}} \sqrt{\frac{1}{\sqrt{\frac{1}{\frac{2\,g\,\rho_0\,x}{B_T}+1}}}} \tag{4.42}$$

Denoting a dimensionless length as $\mathfrak{D} = 2\,g\,\rho_0\,x/B_T$ and after simplification equation (4.42) becomes

$$c = \overbrace{\sqrt{\frac{B_T}{\rho_0}}}^{\text{"standard term"}} \overbrace{\sqrt[4]{\mathfrak{D}+1}}^{\text{correction factor}} \tag{4.43}$$

The time for the sound to travel a small interval distance, dx is

$$d\tau = \frac{dx}{\sqrt{\frac{B_T}{\rho_0}}\sqrt[4]{\mathfrak{D}+1}} \tag{4.44}$$

The time takes for the sound the travel the whole distance is the integration of infinitesimal time

$$t = \int_0^D \frac{dx}{\sqrt{\frac{B_T}{\rho_0}}\sqrt[4]{\mathfrak{D}+1}} \implies t = \int_0^D \frac{\frac{B_T}{2\,g\,\rho_0}\overbrace{\frac{2\,g\,\rho_0\,dx}{B_T}}^{d\mathfrak{D}}}{\sqrt{\frac{B_T}{\rho_0}}\sqrt[4]{\mathfrak{D}+1}} \tag{4.45}$$

Or converting equation (4.45) to dimensionless form and changing the limits of integration as

$$\frac{2\,g\,\sqrt{\rho_0}\,t}{\sqrt{B_T}} = \int_0^{\frac{2\,g\,\rho_0\,D}{B_T}} \frac{d\mathfrak{D}}{\sqrt[4]{\mathfrak{D}+1}} \tag{4.46}$$

The solution of equation (4.46) is

$$\frac{2\,g\,\sqrt{\rho_0}\,t}{\sqrt{B_T}} = \frac{4\left(\frac{2\,g\,\rho_0\,D}{B_T}+1\right)^{\frac{3}{4}}}{3} - \frac{4}{3} = \frac{4}{3}\left(\sqrt[4]{\left(\frac{2\,g\,\rho_0\,D}{B_T}+1\right)^3} - 1\right) \tag{4.47}$$

The time to sound wave to travel, when the compressibility of the liquid is ignored can be written as

$$t = \frac{D}{\sqrt{\frac{B_T}{\rho_0}}} = \frac{D\sqrt{\rho_0}}{\sqrt{B_T}} \Longrightarrow \frac{2\,g\,\sqrt{\rho_0}\,t}{\sqrt{B_T}} = \frac{2\,g\,\rho_0\,D}{B_T} \tag{4.48}$$

The ratio between the corrected estimated to the standard calculation is

$$\text{correction ratio } = \frac{t_{correct}}{t_{standard}} = \frac{\frac{2\,g\,\rho_0\,D}{B_T}}{\frac{4}{3}\left(\sqrt[4]{\left(\frac{2\,g\,\rho_0\,D}{B_T}+1\right)^3}-1\right)} \tag{4.49}$$

It easy to show that

$$\lim_{D\to 0}(\dots) = \lim_{\frac{2\,g\,\rho_0\,D}{B_T}\to 0} \frac{\frac{2\,g\,\rho_0\,D}{B_T}}{\frac{4}{3}\left(\sqrt[4]{\left(\frac{2\,g\,\rho_0\,D}{B_T}+1\right)^3}-1\right)} = 1 \tag{4.50}$$

Figure 4.50 exhibits the correction that has to be applied when the a constant density assumption compared to a variable density model. The example below demonstrates this point.

Example 4.5:
Assume that the deepest point in the ocean off the Marianas Islands (10,923 [m]) is under a pure water at constant temperature. Calculate the time that sound travels to bottom and back when the density is assumed to be constant. Calculate the time that sound travels to bottom and back when the density is assumed to be variable. Compare constant density assumption to variable density to the bulk modulus is considered and value $2.15\ 10^9$ *[N/m].*

SOLUTION

Under the assumption of constant density, and consequently the sound wave velocity (see equation (4.48)), the traveling time is

$$t = \frac{D}{\sqrt{\frac{B_T}{\rho_0}}} = \frac{2\times 10,923}{\sqrt{2.15\,10^9/1000}} = 21846/1459.45 \sim 14.96[sec] \tag{4.V.a}$$

If the variable density model is employed, then the dimensionless parameter is

$$\frac{2\,g\,\rho_0\,D}{B_T} = \frac{2\times 9.81\times 1000\times 10923}{2.15\,10^9} = 0.09967 \sim 0.01 \tag{4.V.b}$$

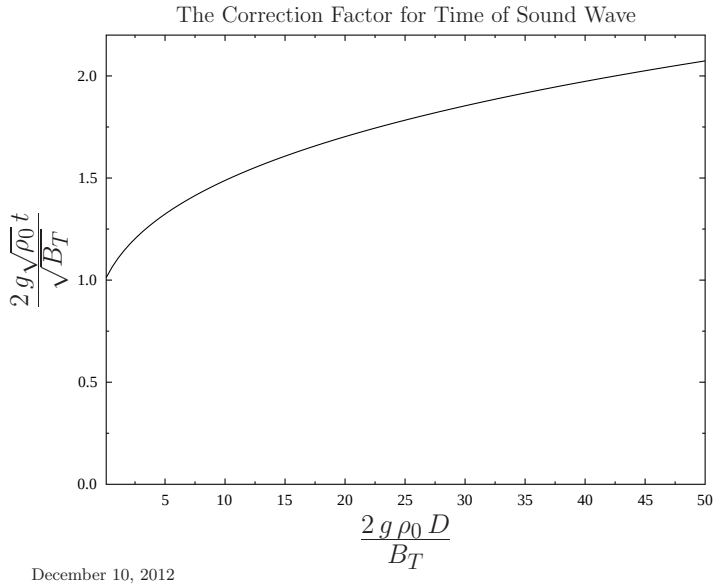


Fig. -4.4. The Correction Factor for Time of Sound Wave.

The correction factor according to equation (4.47)

$$\begin{matrix}\text{correction}\\ \text{factor}\end{matrix}\ \frac{4}{3}\times\Big[(1+0.01)^{(}3/4)-1\Big]\sim 0.098 \qquad \text{(4.V.c)}$$

The significance of these calculations is that error that the deepest place is less deep by about 100 [m].

End Solution

While the ocean calculations results in error of about %1, in geological scale the error is more significant. The next section will be dealing the geological scale.

In summary, the speed of sound in liquids is about 3 to 5 relative to the speed of sound in gases.

4.6 Speed of Sound in Solids

The situation with solids is considerably more complicated, with different speeds in different directions, in different kinds of geometries, and differences between transverse and longitudinal waves. Nevertheless, the speed of sound in solids is larger than in liquids and definitely larger than in gases.

Young's Modulus for a representative value for the bulk modulus for steel is 160 10^9 N $/m^2$.

material	reference	Value [m/sec]
Diamond		12000
Pyrex glass		5640
Steel	longitudinal wave	5790
Steel	transverse shear	3100
Steel	longitudinal wave (extensional wave)	5000
Iron		5130
Aluminum		5100
Brass		4700
Copper		3560
Gold		3240
Lucite		2680
Lead		1322
Rubber		1600

Table -4.3. Solids speed of sound, after Aldred, John, Manual of Sound Recording, London:Fountain Press, 1972

Speed of sound in solid of steel, using a general tabulated value for the bulk modulus, gives a sound speed for structural steel of

$$c = \sqrt{\frac{E}{\rho}} = \sqrt{\frac{160 \times 10^9 N/m^2}{7860 Kg/m^3}} = 4512 m/s \tag{4.51}$$

Compared to one tabulated value the example values for stainless steel lays between the speed for longitudinal and transverse waves.

4.7 Sound Speed in Two Phase Medium

The gas flow in many industrial situations contains other particles. In actuality, there could be more than one speed of sound for two phase flow. Indeed there is double

chocking phenomenon in two phase flow. However, for homogeneous and under certain condition a single velocity can be considered. There can be several models that approached this problem. For simplicity, it assumed that two materials are homogeneously mixed. Topic for none homogeneous mixing are beyond the scope of this book. It further assumed that no heat and mass transfer occurs between the particles. In that case, three extreme cases suggest themselves: the flow is mostly gas with drops of the other phase (liquid or solid), about equal parts of gas and the liquid phase, and liquid with some bubbles. The first case is analyzed.

The equation of state for the gas can be written as

$$P_a = \rho_a \, R \, T_a \tag{4.52}$$

The average density can be expressed as

$$\frac{1}{\rho_m} = \frac{\xi}{\rho_a} + \frac{1-\xi}{\rho_b} \tag{4.53}$$

where $\xi = \frac{\dot{m}_b}{\dot{m}}$ is the mass ratio of the materials. For small value of ξ equation (4.53) can be approximated as

$$\frac{\rho}{\rho_a} = 1 + m \tag{4.54}$$

where $m = \frac{\dot{m}_b}{\dot{m}_a}$ is mass flow rate per gas flow rate.

The gas density can be replaced by equation (4.52) and substituted into equation (4.54)

$$\frac{P}{\rho} = \frac{R}{1+m} T \tag{4.55}$$

A approximation of addition droplets of liquid or dust (solid) results in reduction of R and yet approximate equation similar to ideal gas was obtained. It must noticed that $m = constant$. If the droplets (or the solid particles) can be assumed to have the same velocity as the gas with no heat transfer or fiction between the particles isentropic relation can be assumed as

$$\frac{P}{{\rho_a}^k} = constant \tag{4.56}$$

Assuming that partial pressure of the particles is constant and applying the second law for the mixture yields

$$0 = \overbrace{\frac{m \, C \, dT}{T}}^{droplets} + \overbrace{C_p \, \frac{dT}{T} - R \, \frac{dP}{P}}^{gas} = \frac{(C_p + m \, C) \, dT}{T} - R \, \frac{dP}{P} \tag{4.57}$$

Therefore, the mixture isentropic relationship can be expressed as

$$\frac{P^{\left(\gamma - \frac{1}{\gamma}\right)}}{T} = constant \tag{4.58}$$

where

$$\frac{\gamma - 1}{\gamma} = \frac{R}{C_p + m\,C} \tag{4.59}$$

Recalling that $R = C_p - C_v$ reduces equation (4.59) into

$$\gamma = \frac{C_p + m\,C}{C_v + m\,C} \tag{4.60}$$

In a way the definition of γ was so chosen that effective specific pressure heat and effective specific volumetric heat are $\frac{C_p+mC}{1+m}$ and $\frac{C_v+mC}{1+m}$ respectively. The correction factors for the specific heat is not linear.

Since the equations are the same as before hence the familiar equation for speed of sound can be applied as

Two Phase Sound Speed

$$c = \sqrt{\gamma\, R_{mix}\, T} \tag{4.61}$$

It can be noticed that R_{mix} and γ are smaller than similar variables in a pure gas. Hence, this analysis results in lower speed of sound compared to pure gas. Generally, the velocity of mixtures with large gas component is smaller of the pure gas. For example, the velocity of sound in slightly wet steam can be about one third of the pure steam speed of sound.

Meta

For a mixture of two phases, speed of sound can be expressed as

$$c^2 = \frac{\partial P}{\partial \rho} = \frac{\partial P[f(X)]}{\partial \rho} \tag{4.62}$$

where X is defined as

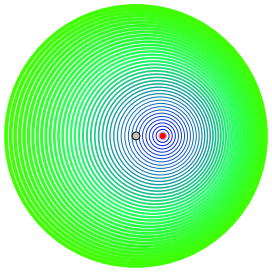
$$X = \frac{s - s_f\,(P_B)}{s_{fg}(P_B)} \tag{4.63}$$

Meta End

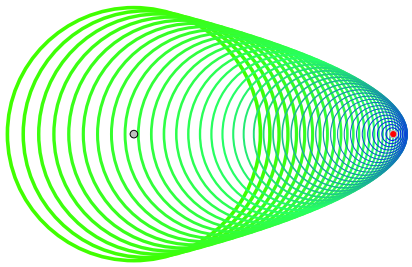
4.8 The Dimensional Effect of the Speed of Sound

What is the significance of the speed of sound? This speed of sound determines what regime the flow will be. In Chapter Dimensional Analsis of Fundamentals of Compressible Flow boo it was shown that Mach number is important parameter. It will be shown later in this Chapter that when Mach number is around 0.25-0.3 a significant change occur in the situation of flow. To demonstrate this point, consider a two dimensional situation where a particle is moving from the left to the right. A particle movement

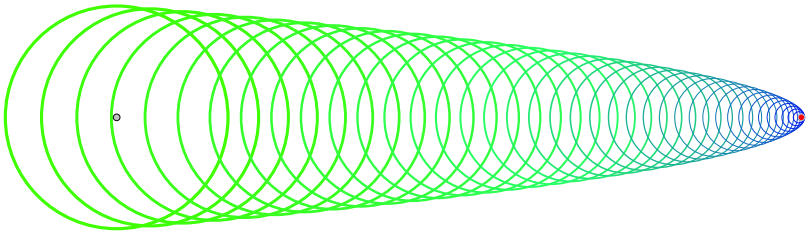
creates a pressure change which travels toward outside in equal speed relative to the particle. Figure 4.5 depicts an object with three different relative velocities. Figure 4.5(a) demonstrates that the whole surroundings is influenced by the object (depicted by red color). While Figure 4.5 (b) that there small zone a head object that is "aware" if the object arriving. In Figure 4.5 (c) the zone that aware of the object is practically zero.



(a) Object travels at 0.005 of the speed of sound



(b) Object travels at 0.05 of the speed of sound



(c) Object travels at 0.15 of the speed of sound

Fig. -4.5. Moving object at three relative velocities. The gray point in the first circle is the initial point the object. The finial point is marked by red circled with gray filled. Notice that the circle line thickness is increase with the time i.e the more green wider circle line thickness. The transition from the blue fresher lines to the green older lines is properly marked.

In fact, when the object velocity is about or larger than the speed of sound then the object arrive to location where the fluid does not aware or informed about the object. The reason that in gas the compressibility plays significant role is because the ratio of the object or fluid velocity compared to speed of sound. In gases the speed of sound is smaller as compare to liquid and defendtly to solid. Hence, gases are media where compressebilty effect must be considered in realtionshp compressebilty. There are some how defined the Mach cone as the shape of object movement approaching to one. This shape has angle and it related to Mach angle. This angle related to Prantle–Meyer angle that will bedefined in the later Chapter.

CHAPTER 5

Isentropic Flow

In this chapter a discussion on a steady state flow through a smooth and continuous area flow rate is presented. A discussion about the flow through a converging–diverging nozzle is also part of this chapter. The isentropic flow models are important because of two main reasons: One, it provides the information about the trends and important parameters. Two, the correction factors can be introduced later to account for deviations from the ideal state.

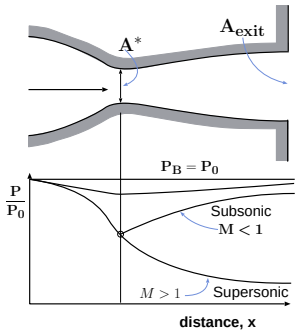


Fig. -5.1. Flow of a compressible substance (gas) through a converging–diverging nozzle.

5.1 Stagnation State for Ideal Gas Model

5.1.1 General Relationship

It is assumed that the flow is one–dimensional. Figure (5.1) describes a gas flow through a converging–diverging nozzle. It has been found that a theoretical state known as the stagnation state is very useful in simplifying the solution and treatment of the flow. The stagnation state is a theoretical state in which the flow is brought into a complete motionless condition in isentropic process without other forces (e.g. gravity force). Several properties that can be represented by this theoretical process which include temperature, pressure, and density et cetera and denoted by the subscript "0."

First, the stagnation temperature is calculated. The energy conservation can

be written as

$$h + \frac{U^2}{2} = h_0 \tag{5.1}$$

Perfect gas is an ideal gas with a constant heat capacity, C_p. For perfect gas equation (5.1) is simplified into

$$C_p T + \frac{U^2}{2} = C_p T_0 \tag{5.2}$$

Again it is common to denote T_0 as the stagnation temperature. Recalling from thermodynamic the relationship for perfect gas

$$R = C_p - C_v \tag{5.3}$$

and denoting $k \equiv C_p \div C_v$ then the thermodynamics relationship obtains the form

$$C_p = \frac{kR}{k-1} \tag{5.4}$$

and where R is a specific constant. Dividing equation (5.2) by $(C_p T)$ yields

$$1 + \frac{U^2}{2C_p T} = \frac{T_0}{T} \tag{5.5}$$

Now, substituting $c^2 = k\,R\,T$ or $T = c^2/k\,R$ equation (5.5) changes into

$$1 + \frac{k\,R\,U^2}{2C_p c^2} = \frac{T_0}{T} \tag{5.6}$$

By utilizing the definition of k by equation (2.24) and inserting it into equation (5.6) yields

$$1 + \frac{k-1}{2}\,\frac{U^2}{c^2} = \frac{T_0}{T} \tag{5.7}$$

It very useful to convert equation (5.6) into a dimensionless form and denote Mach number as the ratio of velocity to speed of sound as

Mach Number Definition

$$M \equiv \frac{U}{c} \tag{5.8}$$

Inserting the definition of Mach number (5.8) into equation (5.7) reads

Isentropic Temperature relationship

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \tag{5.9}$$

The usefulness of Mach number and equation (5.9) can be demonstrated by the following simple example. In this example a gas flows through a tube (see Figure 5.2) of any shape can be expressed as a function of only the stagnation temperature as opposed to the function of the temperatures and velocities.

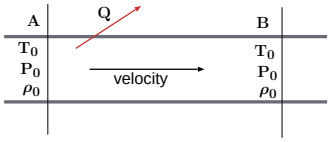


Fig. -5.2. Perfect gas flows through a tube

The definition of the stagnation state provides the advantage of compact writing. For example, writing the energy equation for the tube shown in Figure (5.2) can be reduced to

$$\dot{Q} = C_p \left(T_{0B} - T_{0A}\right) \dot{m} \tag{5.10}$$

The ratio of stagnation pressure to the static pressure can be expressed as the function of the temperature ratio because of the isentropic relationship as

Isentropic Pressure Definition

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{\frac{k}{k-1}} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \tag{5.11}$$

In the same manner the relationship for the density ratio is

Isentropic Density

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{k-1}} = \left(1 + \frac{k-1}{2} M^2\right)^{\frac{1}{k-1}} \tag{5.12}$$

New useful definitions are introduced for the case when $M = 1$ and denoted by superscript "$*$." The special cases of ratio of the star values to stagnation values are dependent only on the heat ratio as the following:

Star Relationship

$$\begin{aligned}
\frac{T^*}{T_0} &= \frac{{c^*}^2}{{c_0}^2} \\
\frac{V_2}{V_1} &= \left(\frac{T_1}{T_2}\right)^{\frac{1}{k-1}} = \left(\frac{\rho_1}{\rho_2}\right) = \left(\frac{P_1}{P_2}\right)^{\frac{1}{k}} \\
\frac{P^*}{P_0} &= \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \\
\frac{\rho^*}{\rho_0} &= \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}
\end{aligned} \tag{5.13}$$

Using all the definitions above relationship between the stagnation properties to star

speed of sound are

$$c^* = \sqrt{k\,R\,\frac{2\,T_0}{k+2}} \tag{5.14}$$

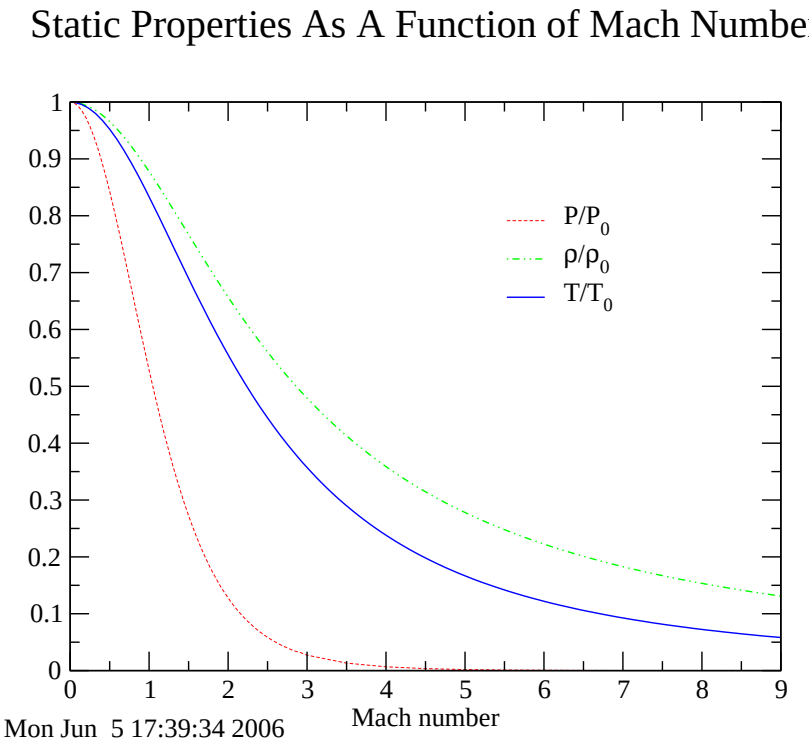


Fig. -5.3. The stagnation properties as a function of the Mach number, $k = 1.4$

calculations/stagnationHTML Title 0.5variableArea:fig:basicStation properties as $f(M)$The stagnation properties as a function of the Mach number, *k=1.4*.

5.1.2 Relationships for Small Mach Number

Even with today's computers a simplified method can reduce the tedious work involved in computational work. In particular, the trends can be examined with analytical methods. It further will be used in the book to examine trends in derived models. It can be noticed that the Mach number involved in the above equations is in a square power. Hence, if an acceptable error is of about %1 then $M < 0.1$ provides the desired range. Further, if a higher power is used, much smaller error results. First it can be noticed that the ratio of temperature to stagnation temperature, $\frac{T}{T_0}$ is provided in power series. Expanding

of the equations according to the binomial expansion of

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots \tag{5.15}$$

will result in the same fashion

$$\frac{P_0}{P} = 1 + \frac{(k-1)M^2}{4} + \frac{kM^4}{8} + \frac{2(2-k)M^6}{48}\cdots \tag{5.16}$$

$$\frac{\rho_0}{\rho} = 1 + \frac{(k-1)M^2}{4} + \frac{kM^4}{8} + \frac{2(2-k)M^6}{48}\cdots \tag{5.17}$$

The pressure difference normalized by the velocity (kinetic energy) as correction factor is

$$\frac{P_0 - P}{\frac{1}{2}\rho U^2} = 1 + \overbrace{\frac{M^2}{4} + \frac{(2-k)M^4}{24} + \cdots}^{compressibility\ correction} \tag{5.18}$$

From the above equation, it can be observed that the correction factor approaches zero when $M \longrightarrow 0$ and then equation (5.18) approaches the standard equation for incompressible flow.

The definition of the star Mach is ratio of the velocity and star speed of sound at $M = 1$.

$$M^* = \frac{U}{c^*} = \sqrt{\frac{k+1}{2}} M \left(1 - \frac{k-1}{4}M^2 + \cdots\right) \tag{5.19}$$

$$\frac{P_0 - P}{P} = \frac{k\,M^2}{2}\left(1 + \frac{M^2}{4} + \cdots\right) \tag{5.20}$$

$$\frac{\rho_0 - \rho}{\rho} = \frac{M^2}{2}\left(1 - \frac{k\,M^2}{4} + \cdots\right) \tag{5.21}$$

The normalized mass rate becomes

$$\frac{\dot{m}}{A} = \sqrt{\frac{k\,{P_0}^2 M^2}{RT_0}}\left(1 + \frac{k-1}{4}M^2 + \cdots\right) \tag{5.22}$$

The ratio of the area to star area is

$$\frac{A}{A^*} = \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}}\left(\frac{1}{M} + \frac{k+1}{4}M + \frac{(3-k)(k+1)}{32}M^3 + \cdots\right) \tag{5.23}$$

5.2 Isentropic Converging-Diverging Flow in Cross Section

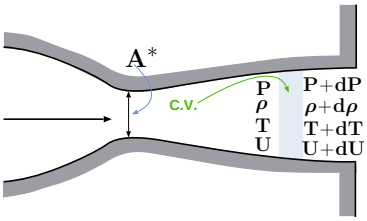


Fig. -5.4. Control volume inside a converging-diverging nozzle.

The important sub case in this chapter is the flow in a converging–diverging nozzle. The control volume is shown in Figure (5.4). There are two models that assume variable area flow: First is isentropic and adiabatic model. Second is isentropic and isothermal model. Clearly, the stagnation temperature, T_0, is constant through the adiabatic flow because there isn't heat transfer. Therefore, the stagnation pressure is also constant through the flow because the flow isentropic. Conversely, in mathematical terms, equation (5.9) and equation (5.11) are the same. If the right hand side is constant for one variable, it is constant for the other. In the same argument, the stagnation density is constant through the flow. Thus, knowing the Mach number or the temperature will provide all that is needed to find the other properties. The only properties that need to be connected are the cross section area and the Mach number. Examination of the relation between properties can then be carried out.

5.2.1 The Properties in the Adiabatic Nozzle

When there is no external work and heat transfer, the energy equation, reads

$$dh + UdU = 0 \tag{5.24}$$

Differentiation of continuity equation, $\rho AU = \dot{m} = constant$, and dividing by the continuity equation reads

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dU}{U} = 0 \tag{5.25}$$

The thermodynamic relationship between the properties can be expressed as

$$T\,ds = dh - \frac{dP}{\rho} \tag{5.26}$$

For isentropic process $ds \equiv 0$ and combining equations (5.24) with (5.26) yields

$$\frac{dP}{\rho} + UdU = 0 \tag{5.27}$$

Differentiation of the equation state (perfect gas), $P = \rho RT$, and dividing the results by the equation of state (ρRT) yields

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{5.28}$$

Obtaining an expression for dU/U from the mass balance equation (5.25) and using it in equation (5.27) reads

$$\frac{dP}{\rho} - U^2 \overbrace{\left[\frac{dA}{A} + \frac{d\rho}{\rho}\right]}^{\frac{dU}{U}} = 0 \tag{5.29}$$

Rearranging equation (5.29) so that the density, ρ, can be replaced by the static pressure, dP/ρ yields

$$\frac{dP}{\rho} = U^2 \left(\frac{dA}{A} + \frac{d\rho}{\rho}\frac{dP}{dP}\right) = U^2 \left(\frac{dA}{A} + \overbrace{\frac{d\rho}{dP}}^{\frac{1}{c^2}} \frac{dP}{\rho}\right) \tag{5.30}$$

Recalling that $dP/d\rho = c^2$ and substitute the speed of sound into equation (5.30) to obtain

$$\frac{dP}{\rho}\left[1 - \left(\frac{U}{c}\right)^2\right] = U^2 \frac{dA}{A} \tag{5.31}$$

Or in a dimensionless form

$$\frac{dP}{\rho}\left(1 - M^2\right) = U^2 \frac{dA}{A} \tag{5.32}$$

Equation (5.32) is a differential equation for the pressure as a function of the cross section area. It is convenient to rearrange equation (5.32) to obtain a variables separation form of

$$dP = \frac{\rho U^2}{A} \frac{dA}{1 - M^2} \tag{5.33}$$

The pressure Mach number relationship

Before going further in the mathematical derivation it is worth looking at the physical meaning of equation (5.33). The term $\rho U^2/A$ is always positive (because all the three terms can be only positive). Now, it can be observed that dP can be positive or negative depending on the dA and Mach number. The meaning of the sign change for the pressure differential is that the pressure can increase or decrease. It can be observed that the critical Mach number is one. If the Mach number is larger than one than dP has opposite sign of dA. If Mach number is smaller than one dP and dA have the same sign. For the subsonic branch $M < 1$ the term $1/(1 - M^2)$ is positive hence

$$dA > 0 \Longrightarrow dP > 0$$
$$dA < 0 \Longrightarrow dP < 0$$

From these observations the trends are similar to those in incompressible fluid. An increase in area results in an increase of the static pressure (converting the dynamic pressure to a static pressure). Conversely, if the area decreases (as a function of x) the pressure decreases. Note that the pressure decrease is larger in compressible flow compared to incompressible flow.

For the supersonic branch $M > 1$, the phenomenon is different. For $M > 1$ the term $1/1 - M^2$ is negative and change the character of the equation.

$$dA > 0 \Rightarrow dP < 0$$
$$dA < 0 \Rightarrow dP > 0$$

This behavior is opposite to incompressible flow behavior.

For the special case of $M = 1$ (sonic flow) the value of the term $1 - M^2 = 0$ thus mathematically $dP \to \infty$ or $dA = 0$. Since physically dP can increase only in a finite amount it must that $dA = 0$. It must also be noted that when $M = 1$ occurs only when $dA = 0$. However, the opposite, not necessarily means that when $dA = 0$ that $M = 1$. In that case, it is possible that $dM = 0$ thus the diverging side is in the subsonic branch and the flow isn't choked.

The relationship between the velocity and the pressure can be observed from equation (5.27) by solving it for dU.

$$dU = -\frac{dP}{P\,U} \tag{5.34}$$

From equation (5.34) it is obvious that dU has an opposite sign to dP (since the term PU is positive). Hence the pressure increases when the velocity decreases and vice versa.

From the speed of sound, one can observe that the density, ρ, increases with pressure and vice versa (see equation (5.35)).

$$d\rho = \frac{1}{c^2} dP \tag{5.35}$$

It can be noted that in the derivations of the above equations (5.34 - 5.35), the equation of state was not used. Thus, the equations are applicable for any gas (perfect or imperfect gas).

The second law (isentropic relationship) dictates that $ds = 0$ and from thermodynamics

$$ds = 0 = C_p \frac{dT}{T} - R \frac{dP}{P}$$

and for perfect gas

$$\frac{dT}{T} = \frac{k-1}{k}\,\frac{dP}{P} \tag{5.36}$$

Thus, the temperature varies according to the same way that pressure does.

The relationship between the Mach number and the temperature can be obtained by utilizing the fact that the process is assumed to be adiabatic $dT_0 = 0$. Differentiation of equation (5.9), the relationship between the temperature and the stagnation temperature becomes

$$dT_0 = 0 = dT\left(1 + \frac{k-1}{2}M^2\right) + T(k-1)MdM \tag{5.37}$$

and simplifying equation (5.37) yields

$$\frac{dT}{T} = -\frac{(k-1)MdM}{1 + \frac{k-1}{2}M^2} \tag{5.38}$$

Relationship Between the Mach Number and Cross Section Area

The equations used in the solution are energy (5.38), second law (5.36), state (5.28), mass (5.25)[1]. Note, equation (5.32) isn't the solution but demonstration of certain properties on the pressure.

The relationship between temperature and the cross section area can be obtained by utilizing the relationship between the pressure and temperature (5.36) and the relationship of pressure and cross section area (5.32). First stage equation (5.38) is combined with equation (5.36) and becomes

$$\frac{(k-1)}{k}\frac{dP}{P} = -\frac{(k-1)MdM}{1 + \frac{k-1}{2}M^2} \tag{5.39}$$

Combining equation (5.39) with equation (5.32) yields

$$\frac{1}{k}\frac{\frac{\rho U^2}{A}\frac{dA}{1-M^2}}{P} = -\frac{MdM}{1 + \frac{k-1}{2}M^2} \tag{5.40}$$

The following identify, $\rho U^2 = kMP$ can be proved as

$$kM^2P = k\overbrace{\frac{U^2}{c^2}}^{M^2}\overbrace{\rho RT}^{P} = k\frac{U^2}{k\,R\,T}\overbrace{\rho\,R\,T}^{P} = \rho U^2 \tag{5.41}$$

Using the identity in equation (5.41) changes equation (5.40) into

$$\frac{dA}{A} = \frac{M^2 - 1}{M\left(1 + \frac{k-1}{2}M^2\right)}dM \tag{5.42}$$

[1]The momentum equation is not used normally in isentropic process, why?

Equation (5.42) is very important because it relates the geometry (area) with the relative velocity (Mach number). In equation (5.42), the factors $M\left(1+\frac{k-1}{2}M^2\right)$ and A are positive regardless of the values of M or A. Therefore, the only factor that affects relationship between the cross area and the Mach number is M^2-1. For $M<1$ the Mach number is varied opposite to the cross section area. In the case of $M>1$ the Mach number increases with the cross section area and vice versa. The special case is when $M=1$ which requires that $dA=0$. This condition imposes that internal[2] flow has to pass a converting–diverging device to obtain supersonic velocity. This minimum area is referred to as "throat."

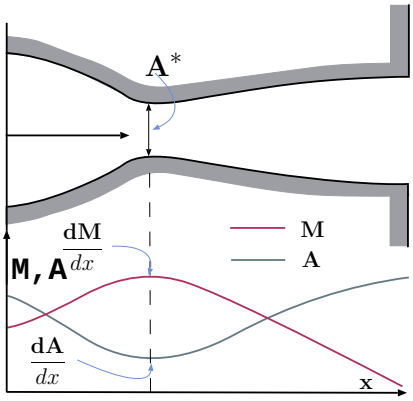


Fig. -5.5. The relationship between the cross section and the Mach number on the subsonic branch

Again, the opposite conclusion that when $dA=0$ implies that $M=1$ is not correct because possibility of $dM=0$. In subsonic flow branch, from the mathematical point of view: on one hand, a decrease of the cross section increases the velocity and the Mach number, on the other hand, an increase of the cross section decreases the velocity and Mach number (see Figure (5.5)).

5.2.2 Isentropic Flow Examples

Example 5.1:
Air is allowed to flow from a reservoir with temperature of 21°C and with pressure of 5[MPa] through a tube. It was measured that air mass flow rate is 1[kg/sec]. At some point on the tube static pressure was measured to be 3[MPa]. Assume that process is isentropic and neglect the velocity at the reservoir, calculate the Mach number, velocity, and the cross section area at that point where the static pressure was measured. Assume that the ratio of specific heat is $k=C_p/C_v=1.4$.

SOLUTION

The stagnation conditions at the reservoir will be maintained throughout the tube because the process is isentropic. Hence the stagnation temperature can be written $T_0=constant$ and $P_0=constant$ and both of them are known (the condition at the reservoir). For the point where the static pressure is known, the Mach number can be calculated by utilizing the pressure ratio. With the known Mach number, the temperature, and velocity can be calculated. Finally, the cross section can be calculated

[2]This condition does not impose any restrictions for external flow. In external flow, an object can be moved in arbitrary speed.

with all these information.

In the point where the static pressure known

$$\bar{P} = \frac{P}{P_0} = \frac{3[MPa]}{5[MPa]} = 0.6$$

From Table (5.2) or from Figure (5.3) or utilizing the enclosed program, Potto-GDC, or simply using the equations shows that

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.88639	0.86420	0.69428	1.0115	0.60000	0.60693	0.53105

With these values the static temperature and the density can be calculated.

$$\begin{aligned} T &= 0.86420338 \times (273 + 21) = 254.076K \\ \rho &= \frac{\rho}{\rho_0}\frac{\overbrace{P_0}^{\rho_0}}{RT_0} = 0.69428839 \times \frac{5 \times 10^6[Pa]}{287.0\left[\frac{J}{kgK}\right] \times 294[K]} \\ &= 41.1416\left[\frac{kg}{m^3}\right] \end{aligned}$$

The velocity at that point is

$$U = M\overbrace{\sqrt{k\,R\,T}}^{c} = 0.88638317 \times \sqrt{1.4 \times 287 \times 254.076} = 283[m/sec]$$

The tube area can be obtained from the mass conservation as

$$A = \frac{\dot{m}}{\rho U} = 8.26 \times 10^{-5}[m^3]$$

For a circular tube the diameter is about 1[cm].

End Solution

Example 5.2:

The Mach number at point A on tube is measured to be $M = 2^3$ and the static pressure is $2[Bar]$[4]. Downstream at point B the pressure was measured to be 1.5[Bar]. Calculate the Mach number at point B under the isentropic flow assumption. Also, estimate the temperature at point B. Assume that the specific heat ratio $k = 1.4$ and assume a perfect gas model.

[4]This pressure is about two atmospheres with temperature of $250[K]$

SOLUTION

With the known Mach number at point A all the ratios of the static properties to total (stagnation) properties can be calculated. Therefore, the stagnation pressure at point A is known and stagnation temperature can be calculated.

At $M = 2$ (supersonic flow) the ratios are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
2.0000	0.55556	0.23005	1.6875	0.12780	0.21567	0.59309

With this information the pressure at point B can be expressed as

$$\frac{P_A}{P_0} = \overbrace{\frac{P_B}{P_0}}^{\substack{\text{from the table} \\ \text{5.2 @ M} = 2}} \times \frac{P_A}{P_B} = 0.12780453 \times \frac{2.0}{1.5} = 0.17040604$$

The corresponding Mach number for this pressure ratio is 1.8137788 and $T_B = 0.60315132$ $\frac{P_B}{P_0} = 0.17040879$. The stagnation temperature can be "bypassed" to calculate the temperature at point $\mathbf{B}$

$$T_B = T_A \times \overbrace{\frac{T_0}{T_A}}^{M=2} \times \overbrace{\frac{T_B}{T_0}}^{M=1.81..} = 250[K] \times \frac{1}{0.55555556} \times 0.60315132 \simeq 271.42[K]$$

End Solution

Example 5.3:

Gas flows through a converging–diverging duct. At point "A" the cross section area is 50 [cm^2] and the Mach number was measured to be 0.4. At point B in the duct the cross section area is 40 [cm^2]. Find the Mach number at point B. Assume that the flow is isentropic and the gas specific heat ratio is 1.4.

SOLUTION

To obtain the Mach number at point B by finding the ratio of the area to the critical area. This relationship can be obtained by

$$\frac{A_B}{A*} = \frac{A_B}{A_A} \times \frac{A_A}{A^*} = \frac{40}{50} \times \overbrace{1.59014}^{\text{from the Table 5.2}} = 1.272112$$

[4] Well, this question is for academic purposes, there is no known way for the author to directly measure the Mach number. The best approximation is by using inserted cone for supersonic flow and measure the oblique shock. Here it is subsonic and this technique is not suitable.

With the value of $\frac{A_B}{A*}$ from the Table (5.2) or from Potto-GDC two solutions can be obtained. The two possible solutions: the first supersonic M = 1.6265306 and second subsonic M = 0.53884934. Both solution are possible and acceptable. The supersonic branch solution is possible only if there where a transition at throat where M=1.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
1.6266	0.65396	0.34585	1.2721	0.22617	0.28772
0.53887	0.94511	0.86838	1.2721	0.82071	1.0440

End Solution

Example 5.4:
Engineer needs to redesign a syringe for medical applications. They complained that the syringe is "hard to push." The engineer analyzes the flow and conclude that the flow is choke. Upon this fact, what engineer should do with the syringe; increase the pushing diameter or decrease the diameter? Explain.

SOLUTION

This problem is a typical to compressible flow in the sense the solution is opposite the regular intuition. The diameter should be decreased. The pressure in the choke flow in the syringe is past the critical pressure ratio. Hence, the force is a function of the cross area of the syringe. So, to decrease the force one should decrease the area.

End Solution

5.2.3 Mass Flow Rate (Number)

One of the important engineering parameters is the mass flow rate which for ideal gas is

$$\dot{m} = \rho U A = \frac{P}{RT} U A \tag{5.43}$$

This parameter is studied here, to examine the maximum flow rate and to see what is the effect of the compressibility on the flow rate. The area ratio as a function of the Mach number needed to be established, specifically and explicitly the relationship for the chocked flow. The area ratio is defined as the ratio of the cross section at any point to the throat area (the narrow area). It is convenient to rearrange the equation (5.43) to be expressed in terms of the stagnation properties as

$$\frac{\dot{m}}{A} = \frac{P}{P_0} \frac{P_0 U}{\sqrt{k\,R\,T}} \sqrt{\frac{k}{R}} \sqrt{\frac{T_0}{T}} \frac{1}{\sqrt{T_0}} = \frac{P_0}{\sqrt{T_0}} M \sqrt{\frac{k}{R}} \overbrace{\frac{P}{P_0} \sqrt{\frac{T_0}{T}}}^{f(M,k)} \tag{5.44}$$

Expressing the temperature in terms of Mach number in equation (5.44) results in

Mass Flow Rate

$$\frac{\dot{m}}{A} = \left(\frac{kMP_0}{\sqrt{k\,R\,T_0}}\right)\left(1+\frac{k-1}{2}M^2\right)^{-\frac{k+1}{2(k-1)}} \tag{5.45}$$

It can be noted that equation (5.45) holds everywhere in the converging-diverging duct and this statement also true for the throat. The throat area can be denoted as by A^*. It can be noticed that at the throat when the flow is chocked or in other words $M = 1$ and that the stagnation conditions (i.e. temperature, pressure) do not change. Hence equation (5.45) obtained the form

$$\frac{\dot{m}}{A^*} = \left(\frac{\sqrt{k}P_0}{\sqrt{RT_0}}\right)\left(1+\frac{k-1}{2}\right)^{-\frac{k+1}{2(k-1)}} \tag{5.46}$$

Since the mass flow rate is constant in the duct, dividing equations (5.46) by equation (5.45) yields

Mass Flow Rate Ratio

$$\frac{A}{A^*} = \frac{1}{M}\left(\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2(k-1)}} \tag{5.47}$$

Equation (5.47) relates the Mach number at any point to the cross section area ratio.

The maximum flow rate can be expressed either by taking the derivative of equation (5.46) in with respect to M and equating to zero. Carrying this calculation results at $M = 1$.

$$\left(\frac{\dot{m}}{A^*}\right)_{max}\frac{P_0}{\sqrt{T_0}} = \sqrt{\frac{k}{R}}\left(\frac{k+1}{2}\right)^{-\frac{k+1}{2(k-1)}} \tag{5.48}$$

For specific heat ratio, $k = 1.4$

$$\left(\frac{\dot{m}}{A^*}\right)_{max}\frac{P_0}{\sqrt{T_0}} \sim \frac{0.68473}{\sqrt{R}} \tag{5.49}$$

The maximum flow rate for air ($R = 287 j/kgK$) becomes,

$$\frac{\dot{m}\sqrt{T_0}}{A^*P_0} = 0.040418 \tag{5.50}$$

Equation (5.50) is known as Fliegner's Formula on the name of one of the first engineers who observed experimentally the choking phenomenon. It can be noticed that Fliegner's equation can lead to definition of the Fliegner's Number.

$$\frac{\dot{m}\sqrt{T_0}}{A^*P_0} = \frac{\dot{m}\overbrace{\sqrt{k\,R\,T_0}}^{c_0}}{\sqrt{k\,R}A^*P_0} = \frac{1}{\sqrt{R}}\overbrace{\frac{\dot{m}\,c_0}{A^*P_0}}^{Fn}\frac{1}{\sqrt{k}} \tag{5.51}$$

The definition of Fliegner's number (Fn) is

$$\boldsymbol{Fn} \equiv \frac{\sqrt{R}\,\dot{m}\,c_0}{\sqrt{R}\,A^*\,P_0} \tag{5.52}$$

Utilizing Fliegner's number definition and substituting it into equation (5.46) results in

Fliegner's Number

$$\boldsymbol{Fn} = k\,M\left(1+\frac{k-1}{2}M^2\right)^{-\frac{k+1}{2(k-1)}} \tag{5.53}$$

and the maximum point for Fn at $M=1$ is

$$\boldsymbol{Fn} = k\left(\frac{k+1}{2}\right)^{-\frac{k+1}{2(k-1)}} \tag{5.54}$$

Example 5.5:
Why $\boldsymbol{Fn}$ *is zero at Mach equal to zero? Prove Fliegner number,* $\boldsymbol{Fn}$ *is maximum at* $M=1$.

"Naughty Professor" Problems in Isentropic Flow

To explain the material better some instructors invented problems, which have mostly academic purpose, (see for example, Shapiro (problem 4.5)). While these problems have a limit applicability in reality, they have substantial academic value and therefore presented here. The situation where the mass flow rate per area given with one of the stagnation properties and one of the static properties, e.g. P_0 and T or T_0 and P present difficulty for the calculations. The use of the regular isentropic Table is not possible because there isn't variable represent this kind problems. For this kind of problems a new Table was constructed and present here[5].

The case of T_0 and P

This case considered to be the simplest and is presented here first. Using energy equation (5.9) and substituting for Mach number $M=\dot{m}/(A\,\rho\,c)$ results in

$$\frac{T_0}{T} = 1+\frac{k-1}{2}\left(\frac{\dot{m}}{A\rho c}\right)^2 \tag{5.55}$$

Multiplying equation (5.55) by $T\,\rho^2$ result in

$$T_0\rho^2 = \overbrace{T\rho}^{\frac{P}{R}}\,\rho + \overbrace{\left(\frac{T}{c^2}\right)}^{\frac{1}{kR}}\frac{k-1}{2}\left(\frac{\dot{m}}{A}\right)^2 \tag{5.56}$$

[5]Since version 0.4.4 of this book.

And further Rearranging equation (5.56) transformed it into

$$\rho^2 = \frac{P\rho}{T_0\,R} + \frac{k-1}{2\,k\,R\,T_0}\left(\frac{\dot{m}}{A}\right)^2 \tag{5.57}$$

Equation (5.57) is quadratic equation for density, ρ when all other variables are known. It is convenient to change it into

$$\rho^2 - \frac{P\,\rho}{T_0\,R} - \frac{k-1}{2\,k\,R\,T_0}\left(\frac{\dot{m}}{A}\right)^2 = 0 \tag{5.58}$$

The only physical solution is when the density is positive and thus the only solution is

$$\rho = \frac{1}{2}\left[\frac{P}{R\,T_0} + \sqrt{\left(\frac{P}{RT_0}\right)^2 + \underbrace{2\frac{k-1}{k\,R\,T_0}\left(\frac{\dot{m}}{A}\right)^2}_{\hookrightarrow(M\to 0)\to 0}}\right] \tag{5.59}$$

For almost incompressible flow the density is reduced and the familiar form of perfect gas model is seen since stagnation temperature is approaching the static temperature for very small Mach number ($\rho = \frac{P}{R\,T_0}$). In other words, the terms for the terms over the under–brace approaches zero when the flow rate (Mach number) is very small.

It is convenient to denote a new dimensionless density as

$$\hat{\boldsymbol{\rho}} = \frac{\rho}{\frac{p}{R\,T_0}} = \frac{\rho\,R\,T_0}{P} = \frac{1}{\overline{T}} \tag{5.60}$$

With this new definition equation (5.59) is transformed into

$$\hat{\boldsymbol{\rho}} = \frac{1}{2}\left(1 + \sqrt{1 + 2\frac{(k-1)\,R\,T_0}{k\,P^2}\left(\frac{\dot{m}}{A}\right)^2}\right) \tag{5.61}$$

The dimensionless density now is related to a dimensionless group that is a function of $\boldsymbol{Fn}$ number and Mach number only! Thus, this dimensionless group is function of Mach number only.

$$\frac{R\,T_0}{P^2}\left(\frac{\dot{m}}{A}\right)^2 = \frac{1}{k}\overbrace{\frac{{c_0}^2}{{P_0}^2}\left(\frac{\dot{m}}{A^*}\right)^2}^{\boldsymbol{Fn}^2}\overbrace{\left(\frac{A^*}{A}\right)^2\left(\frac{P_0}{P}\right)^2}^{\frac{A^*P_0}{AP}=f(M)} \tag{5.62}$$

Thus,

$$\frac{R\,T_0}{P^2}\left(\frac{\dot{m}}{A}\right)^2 = \frac{\boldsymbol{Fn}^2}{k}\left(\frac{A^*P_0}{AP}\right)^2 \tag{5.63}$$

Hence, the dimensionless density is

$$\hat{\boldsymbol{\rho}} = \frac{1}{2}\left(1 + \sqrt{1 + 2\frac{(k-1)Fn^2}{k^2}\left(\frac{A^* P_0}{AP}\right)^2}\right) \tag{5.64}$$

Again notice that the right hand side of equation (5.64) is only function of Mach number (well, also the specific heat, k). And the values of $\frac{AP}{A^* P_0}$ were tabulated in Table (5.2) and Fn is tabulated in the next Table (5.1). Thus, the problems is reduced to finding tabulated values.

Example 5.6:
The pitot tube measured the temperature of a flow which was found to be 300° C. The static pressure was measured to be 2 [Bar]. The flow rate is 1 [kg/sec] and area of the conduct is 0.001 [m^2]. Calculate the Mach number, the velocity of the stream, and stagnation pressure. Assume perfect gas model with k=1.42.

SOLUTION

This exactly the case discussed above in which the the ratio of mass flow rate to the area is given along with the stagnation temperature and static pressure. Utilizing equation (5.63) will provide the solution.

$$\frac{RT_0}{P^2}\left(\frac{\dot{m}}{A}\right)^2 = \frac{287 \times 373}{200,000^2} \times \left(\frac{1}{0.001}\right)^2 = 2.676275 \tag{5.VI.a}$$

According to Table 5.1 the Mach number is about $M = 0.74\cdots$ (the exact number does not appear here demonstrate the simplicity of the solution). The Velocity can be obtained from the

$$U = M\, c = M\sqrt{k\, R\, T} \tag{5.VI.b}$$

The only unknown the equation (5.VI.b) is the temperature. However, the temperature can be obtained from knowing the Mach number with the "regular" table. Utilizing the regular table or Potto GDC one obtained.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.74000	0.89686	0.77169	1.0677	0.69210	0.73898	0.54281

The temperature is then

$$T = (287 + 300) \times 0.89686 \sim 526.45K \sim 239.4^{\circ}\mathrm{C} \tag{5.VI.c}$$

Hence the velocity is

$$U = 0.74 \times \sqrt{1.42 \times 287 \times 526.45} \sim 342.76[m/sec] \tag{5.VI.d}$$

In the same way the static pressure is

$$P_0 = P \Big/ \frac{P}{P_0} \sim 2/0.692 \sim 2.89[Bar] \tag{5.VI.e}$$

The usage of Table 5.1 is only approximation and the exact value can be obtained utilizing Potto GDC.

End Solution

The case of P_0 and T

A similar problem can be described for the case of stagnation pressure, P_0, and static temperature, T.

First, it is shown that the dimensionless group is a function of Mach number only (well, again the specific heat ratio, k also).

$$\frac{RT}{{P_0}^2}\left(\frac{\dot{m}}{A}\right)^2 = \frac{Fn^2}{k}\left(\frac{A^* P_0}{AP}\right)^2\left(\frac{T}{T_0}\right)\left(\frac{P_0}{P}\right)^2 \tag{5.65}$$

It can be noticed that

$$\frac{Fn^2}{k} = \left(\frac{T}{T_0}\right)\left(\frac{P_0}{P}\right)^2 \tag{5.66}$$

Thus equation (5.65) became

$$\frac{RT}{{P_0}^2}\left(\frac{\dot{m}}{A}\right)^2 = \left(\frac{A^* P_0}{A P}\right)^2 \tag{5.67}$$

The right hand side is tabulated in the "regular" isentropic Table such (5.2). This example shows how a dimensional analysis is used to solve a problems without actually solving any equations. The actual solution of the equation is left as exercise (this example under construction). What is the legitimacy of this method? The explanation simply based the previous experience in which for a given ratio of area or pressure ratio (etcetera) determines the Mach number. Based on the same arguments, if it was shown that a group of parameters depends only Mach number than the Mach is determined by this group.

The method of solution for given these parameters is by calculating the $\frac{PA}{P_0 A^*}$ and then using the table to find the corresponding Mach number.

Example 5.7:
Calculate the Mach number for flow with given stagnation pressure of 2 [Bar] and 27° C. It is given that the mass flow rate is 1 [kg/sec] and the cross section area is 0.01[m^2]. Assume that the specific heat ratios, k =1.4.

SOLUTION

To solve this problem, the ratio in equation (5.67) has to be found.

$$\left(\frac{A^* P_0}{A P}\right)^2 = \frac{R T}{{P_0}^2}\left(\frac{\dot{m}}{A}\right)^2 = \frac{287 \times 300}{200000^2}\left(\frac{1}{0.01}\right)^2 \sim 0.021525 \qquad \text{(5.VII.a)}$$

This mean that $\frac{A^* P_0}{A P} \sim 0.1467$. In the table it translate into

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.08486	0.99856	0.99641	6.8487	0.99497	6.8143	2.8679

End Solution

The case of ρ_0 and T or P

The last case sometimes referred to as the "naughty professor's question" case dealt here is when the stagnation density given with the static temperature/pressure. First, the dimensionless approach is used and later analytical method is discussed (under construction).

$$\frac{1}{R\,\rho_0\,P}\left(\frac{\dot{m}}{A}\right)^2 = \frac{\overbrace{k R T_0}^{{c_0}^2}}{k\,R\,P_0\,P_0\frac{P}{P_0}}\left(\frac{\dot{m}}{A}\right)^2 = \frac{{c_0}^2}{k\,R\,{P_0}^2\frac{P}{P_0}}\left(\frac{\dot{m}}{A}\right)^2 = \frac{Fn^2}{k}\left(\frac{P_0}{P}\right) \qquad (5.68)$$

The last case dealt here is of the stagnation density with static pressure and the following is dimensionless group

$$\frac{1}{R\,{\rho_0}^2\,T}\left(\frac{\dot{m}}{A}\right)^2 = \frac{\overbrace{k\,R\,T_0}^{{c_0}^2}\,T_0}{k\,R\,{P_0}^2\,T}\left(\frac{\dot{m}}{A}\right)^2 = \frac{{c_0}^2 T_0}{k R {P_0}^2 T}\left(\frac{\dot{m}}{A}\right)^2 = \frac{Fn^2}{k}\left(\frac{T_0}{T}\right) \qquad (5.69)$$

It was hidden in the derivations/explanations of the above analysis didn't explicitly state under what conditions these analysis is correct. Unfortunately, not all the analysis valid for the same conditions and is as the regular "isentropic" Table, (5.2). The heat/temperature part is valid for **enough** adiabatic condition while the pressure condition requires also isentropic process. All the above conditions/situations require to have the perfect gas model as the equation of state. For example the first "naughty professor" question is sufficient that process is adiabatic only (T_0, P, mass flow rate per area.).

Table -5.1. Fliegner's number and other parameters as a function of Mach number

$\mathbf{M}$	$\mathbf{Fn}$	$\hat{\rho}$	$\left(\frac{\mathbf{P_0 A^*}}{\mathbf{AP}}\right)^{\mathbf{2}}$	$\frac{\mathbf{RT_0}}{\mathbf{P^2}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{R}\rho_0\mathbf{P}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{R}\rho_0{}^{\mathbf{2}}\mathbf{T}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$
0.00	$1.4E{-}06$	1.000	0.0	0.0	0.0	0.0
0.05	0.070106	1.000	0.00747	$2.62E{-}05$	0.00352	0.00351
0.10	0.14084	1.000	0.029920	0.000424	0.014268	0.014197
0.20	0.28677	1.001	0.12039	0.00707	0.060404	0.059212
0.21	0.30185	1.001	0.13284	0.00865	0.067111	0.065654
0.22	0.31703	1.001	0.14592	0.010476	0.074254	0.072487
0.23	0.33233	1.002	0.15963	0.012593	0.081847	0.079722
0.24	0.34775	1.002	0.17397	0.015027	0.089910	0.087372
0.25	0.36329	1.003	0.18896	0.017813	0.098460	0.095449
0.26	0.37896	1.003	0.20458	0.020986	0.10752	0.10397
0.27	0.39478	1.003	0.22085	0.024585	0.11710	0.11294
0.28	0.41073	1.004	0.23777	0.028651	0.12724	0.12239
0.29	0.42683	1.005	0.25535	0.033229	0.13796	0.13232
0.30	0.44309	1.005	0.27358	0.038365	0.14927	0.14276
0.31	0.45951	1.006	0.29247	0.044110	0.16121	0.15372
0.32	0.47609	1.007	0.31203	0.050518	0.17381	0.16522
0.33	0.49285	1.008	0.33226	0.057647	0.18709	0.17728
0.34	0.50978	1.009	0.35316	0.065557	0.20109	0.18992
0.35	0.52690	1.011	0.37474	0.074314	0.21584	0.20316
0.36	0.54422	1.012	0.39701	0.083989	0.23137	0.21703
0.37	0.56172	1.013	0.41997	0.094654	0.24773	0.23155
0.38	0.57944	1.015	0.44363	0.10639	0.26495	0.24674
0.39	0.59736	1.017	0.46798	0.11928	0.28307	0.26264
0.40	0.61550	1.019	0.49305	0.13342	0.30214	0.27926
0.41	0.63386	1.021	0.51882	0.14889	0.32220	0.29663

Table -5.1. Fliegner's number and other parameters as function of Mach number (continue)

$\mathbf{M}$	$\mathbf{Fn}$	$\hat{\rho}$	$\left(\frac{\mathbf{P_0 A^*}}{\mathbf{AP}}\right)^{\mathbf{2}}$	$\frac{\mathbf{RT_0}}{\mathbf{P^2}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{R}\rho_0\mathbf{P}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{R}\rho_0{}^{\mathbf{2}}\mathbf{T}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$
0.42	0.65246	1.023	0.54531	0.16581	0.34330	0.31480
0.43	0.67129	1.026	0.57253	0.18428	0.36550	0.33378
0.44	0.69036	1.028	0.60047	0.20442	0.38884	0.35361
0.45	0.70969	1.031	0.62915	0.22634	0.41338	0.37432
0.46	0.72927	1.035	0.65857	0.25018	0.43919	0.39596
0.47	0.74912	1.038	0.68875	0.27608	0.46633	0.41855
0.48	0.76924	1.042	0.71967	0.30418	0.49485	0.44215
0.49	0.78965	1.046	0.75136	0.33465	0.52485	0.46677
0.50	0.81034	1.050	0.78382	0.36764	0.55637	0.49249
0.51	0.83132	1.055	0.81706	0.40333	0.58952	0.51932
0.52	0.85261	1.060	0.85107	0.44192	0.62436	0.54733
0.53	0.87421	1.065	0.88588	0.48360	0.66098	0.57656
0.54	0.89613	1.071	0.92149	0.52858	0.69948	0.60706
0.55	0.91838	1.077	0.95791	0.57709	0.73995	0.63889
0.56	0.94096	1.083	0.99514	0.62936	0.78250	0.67210
0.57	0.96389	1.090	1.033	0.68565	0.82722	0.70675
0.58	0.98717	1.097	1.072	0.74624	0.87424	0.74290
0.59	1.011	1.105	1.112	0.81139	0.92366	0.78062
0.60	1.035	1.113	1.152	0.88142	0.97562	0.81996
0.61	1.059	1.122	1.194	0.95665	1.030	0.86101
0.62	1.084	1.131	1.236	1.037	1.088	0.90382
0.63	1.109	1.141	1.279	1.124	1.148	0.94848
0.64	1.135	1.151	1.323	1.217	1.212	0.99507
0.65	1.161	1.162	1.368	1.317	1.278	1.044
0.66	1.187	1.173	1.414	1.423	1.349	1.094

Table -5.1. Fliegner's number and other parameters as function of Mach number (continue)

$\mathbf{M}$	$\mathbf{Fn}$	$\hat{\rho}$	$\left(\frac{\mathbf{P_0 A^*}}{\mathbf{AP}}\right)^{\mathbf{2}}$	$\frac{\mathbf{RT_0}}{\mathbf{P^2}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{R}\rho_0\mathbf{P}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{R}\rho_0{}^{\mathbf{2}}\mathbf{T}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$
0.67	1.214	1.185	1.461	1.538	1.422	1.147
0.68	1.241	1.198	1.508	1.660	1.500	1.202
0.69	1.269	1.211	1.557	1.791	1.582	1.260
0.70	1.297	1.225	1.607	1.931	1.667	1.320
0.71	1.326	1.240	1.657	2.081	1.758	1.382
0.72	1.355	1.255	1.708	2.241	1.853	1.448
0.73	1.385	1.271	1.761	2.412	1.953	1.516
0.74	1.415	1.288	1.814	2.595	2.058	1.587
0.75	1.446	1.305	1.869	2.790	2.168	1.661
0.76	1.477	1.324	1.924	2.998	2.284	1.738
0.77	1.509	1.343	1.980	3.220	2.407	1.819
0.78	1.541	1.362	2.038	3.457	2.536	1.903
0.79	1.574	1.383	2.096	3.709	2.671	1.991
0.80	1.607	1.405	2.156	3.979	2.813	2.082
0.81	1.642	1.427	2.216	4.266	2.963	2.177
0.82	1.676	1.450	2.278	4.571	3.121	2.277
0.83	1.712	1.474	2.340	4.897	3.287	2.381
0.84	1.747	1.500	2.404	5.244	3.462	2.489
0.85	1.784	1.526	2.469	5.613	3.646	2.602
0.86	1.821	1.553	2.535	6.006	3.840	2.720
0.87	1.859	1.581	2.602	6.424	4.043	2.842
0.88	1.898	1.610	2.670	6.869	4.258	2.971
0.89	1.937	1.640	2.740	7.342	4.484	3.104
0.90	1.977	1.671	2.810	7.846	4.721	3.244
0.91	2.018	1.703	2.882	8.381	4.972	3.389

Table -5.1. Fliegner's number and other parameters as function of Mach number (continue)

$\mathbf{M}$	$\mathbf{Fn}$	$\hat{\rho}$	$\left(\frac{\mathbf{P_0 A^*}}{\mathbf{AP}}\right)^{\mathbf{2}}$	$\frac{\mathbf{RT_0}}{\mathbf{P^2}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{R}\rho_0\mathbf{P}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{R}\rho_0{}^{\mathbf{2}}\mathbf{T}}\left(\frac{\dot{\mathbf{m}}}{\mathbf{A}}\right)^{\mathbf{2}}$
0.92	2.059	1.736	2.955	8.949	5.235	3.541
0.93	2.101	1.771	3.029	9.554	5.513	3.699
0.94	2.144	1.806	3.105	10.20	5.805	3.865
0.95	2.188	1.843	3.181	10.88	6.112	4.037
0.96	2.233	1.881	3.259	11.60	6.436	4.217
0.97	2.278	1.920	3.338	12.37	6.777	4.404
0.98	2.324	1.961	3.419	13.19	7.136	4.600
0.99	2.371	2.003	3.500	14.06	7.515	4.804
1.00	2.419	2.046	3.583	14.98	7.913	5.016

Example 5.8:
A gas flows in the tube with mass flow rate of 0.1 [kg/sec] and tube cross section is 0.001[m^2]. The temperature at chamber supplying the pressure to tube is 27°C. At some point the static pressure was measured to be 1.5[Bar]. Calculate for that point the Mach number, the velocity, and the stagnation pressure. Assume that the process is isentropic, $k = 1.3$, $R = 287[j/kgK]$.

SOLUTION

The first thing that need to be done is to find the mass flow per area and it is

$$\frac{\dot{m}}{A} = 0.1/0.001 = 100.0[kg/sec/m^2]$$

It can be noticed that the total temperature is $300K$ and the static pressure is 1.5[Bar]. The solution is based on section equations (5.59) through (5.64). It is fortunate that Potto-GDC exist and it can be just plug into it and it provide that

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.17124	0.99562	0.98548	3.4757	0.98116	3.4102	1.5392

The velocity can be calculated as

$$U = M\,c = \sqrt{k\,R\,T}\,M = 0.17 \times \sqrt{1.3 \times 287 \times 300\times} \sim 56.87[m/sec]$$

The stagnation pressure is

$$P_0 = \frac{P}{P/P_0} = 1.5/0.98116 = 1.5288[Bar]$$

End Solution

Flow with pressure losses

The expression for the mass flow rate (5.45) is appropriate regardless the flow is isentropic or adiabatic. That expression was derived based on the theoretical total pressure and temperature (Mach number) which does not based on the considerations whether the flow is isentropic or adiabatic. In the same manner the definition of A^* referred to the theoretical minimum area ("throat area") if the flow continues to flow in an isentropic manner. Clearly, in a case where the flow isn't isentropic or adiabatic the total pressure and the total temperature will change (due to friction, and heat transfer). A constant flow rate requires that $\dot{m}_A = \dot{m}_B$. Denoting subscript A for one point and subscript B for another point mass equation (5.46) can be equated as

$$\left(\frac{kP_0A^*}{RT_0}\right)\left(1+\frac{k-1}{2}M^2\right)^{-\frac{k-1}{2(k-1)}} = constant \tag{5.70}$$

From equation (5.70), it is clear that the function $f(P_0, T_0, A^*) = constant$. There are two possible models that can be used to simplify the calculations. The first model for neglected heat transfer (adiabatic) flow and in which the total temperature remained constant (Fanno flow like). The second model which there is significant heat transfer but insignificant pressure loss (Rayleigh flow like).

If the mass flow rate is constant at any point on the tube (no mass loss occur) then

$$\dot{m} = A^*\sqrt{\frac{k}{RT_0}\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}}}P_0 \tag{5.71}$$

For adiabatic flow, comparison of mass flow rate at point A and point B leads to

$$P_0A^*|_A = P_0A^*|_B$$

$$\rightsquigarrow \frac{P_0|_A}{P_0|_B} = \frac{A^*|_A}{A^*|_B} \tag{5.72}$$

And utilizing the equality of $A^* = \frac{A^*}{A}A$ leads to

$$\frac{P_0|_A}{P_0|_B} = \frac{\frac{A}{A^*}\big|_{M_A}}{\frac{A}{A^*}\big|_{M_B}}\frac{A|_A}{A|_B} \tag{5.73}$$

For a flow with a constant stagnation pressure (frictionless flow) and non adiabatic flow reads

$$\frac{T_0|_A}{T_0|_B} = \left[\frac{\frac{B}{A^*}\big|_{M_B}}{\frac{A}{A^*}\big|_{M_A}} \frac{A|_B}{A|_A} \right]^2 \tag{5.74}$$

Example 5.9:
At point A of the tube the pressure is $3[Bar]$, Mach number is 2.5, and the duct section area is 0.01$[m^2]$. Downstream at exit of tube, point B, the cross section area is 0.015$[m^2]$ and Mach number is 1.5. Assume no mass lost and adiabatic steady state flow, calculated the total pressure lost.

SOLUTION

Both Mach numbers are known, thus the area ratios can be calculated. The total pressure can be calculated because the Mach number and static pressure are known. With these information, and utilizing equation (5.73) the stagnation pressure at point B can be obtained.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
1.5000	0.68966	0.39498	1.1762	0.27240	0.32039	0.55401
2.5000	0.44444	0.13169	2.6367	0.05853	0.15432	0.62693

First, the stagnation at point A is obtained from Table (5.2) as

$$P_0|_A = \left. \frac{P}{\underbrace{\left(\frac{P}{P_0}\right)}_{M=2.5}} \right|_A = \frac{3}{0.058527663} = 51.25781291[Bar]$$

by utilizing equation (5.73) provides

$$P_0|_B = 51.25781291 \times \frac{1.1761671}{2.6367187} \times \frac{0.01}{0.015} \approx 15.243[Bar]$$

Hence

$$P_0\,|_A - P_0|_B = 51.257 - 15.243 = 36.013[Bar]$$

Note that the large total pressure loss is much larger than the static pressure loss (Pressure point B the pressure is $0.27240307 \times 15.243 = 4.146$[Bar]).

End Solution

5.3 Isentropic Tables

Table -5.2. Isentropic Table $k = 1.4$

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.00	1.00000	1.00000	$5.8E{+}5$	1.0000	$5.8E+5$	$2.4E{+}5$
0.05	0.99950	0.99875	11.59	0.99825	11.57	4.838
0.10	0.99800	0.99502	5.822	0.99303	5.781	2.443
0.20	0.99206	0.98028	2.964	0.97250	2.882	1.268
0.30	0.98232	0.95638	2.035	0.93947	1.912	0.89699
0.40	0.96899	0.92427	1.590	0.89561	1.424	0.72632
0.50	0.95238	0.88517	1.340	0.84302	1.130	0.63535
0.60	0.93284	0.84045	1.188	0.78400	0.93155	0.58377
0.70	0.91075	0.79158	1.094	0.72093	0.78896	0.55425
0.80	0.88652	0.73999	1.038	0.65602	0.68110	0.53807
0.90	0.86059	0.68704	1.009	0.59126	0.59650	0.53039
0.95	0.00328	1.061	1.002	1.044	0.95781	1.017
0.96	0.00206	1.049	1.001	1.035	0.96633	1.013
0.97	0.00113	1.036	1.001	1.026	0.97481	1.01
0.98	0.000495	1.024	1.0	1.017	0.98325	1.007
0.99	0.000121	1.012	1.0	1.008	0.99165	1.003
1.00	0.83333	0.63394	1.000	0.52828	0.52828	0.52828
1.1	0.80515	0.58170	1.008	0.46835	0.47207	0.52989
1.2	0.77640	0.53114	1.030	0.41238	0.42493	0.53399
1.3	0.74738	0.48290	1.066	0.36091	0.38484	0.53974
1.4	0.71839	0.43742	1.115	0.31424	0.35036	0.54655
1.5	0.68966	0.39498	1.176	0.27240	0.32039	0.55401
1.6	0.66138	0.35573	1.250	0.23527	0.29414	0.56182
1.7	0.63371	0.31969	1.338	0.20259	0.27099	0.56976
1.8	0.60680	0.28682	1.439	0.17404	0.25044	0.57768

Table -5.2. Isentropic Table k=1.4 *(continue)*

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
1.9	0.58072	0.25699	1.555	0.14924	0.23211	0.58549
2.0	0.55556	0.23005	1.688	0.12780	0.21567	0.59309
2.5	0.44444	0.13169	2.637	0.058528	0.15432	0.62693
3.0	0.35714	0.076226	4.235	0.027224	0.11528	0.65326
3.5	0.28986	0.045233	6.790	0.013111	0.089018	0.67320
4.0	0.23810	0.027662	10.72	0.00659	0.070595	0.68830
4.5	0.19802	0.017449	16.56	0.00346	0.057227	0.69983
5.0	0.16667	0.011340	25.00	0.00189	0.047251	0.70876
5.5	0.14184	0.00758	36.87	0.00107	0.039628	0.71578
6.0	0.12195	0.00519	53.18	0.000633	0.033682	0.72136
6.5	0.10582	0.00364	75.13	0.000385	0.028962	0.72586
7.0	0.092593	0.00261	$1.0E+2$	0.000242	0.025156	0.72953
7.5	0.081633	0.00190	$1.4E+2$	0.000155	0.022046	0.73257
8.0	0.072464	0.00141	$1.9E+2$	0.000102	0.019473	0.73510
8.5	0.064725	0.00107	$2.5E+2$	$6.90E-5$	0.017321	0.73723
9.0	0.058140	0.000815	$3.3E+2$	$4.74E-5$	0.015504	0.73903
9.5	0.052493	0.000631	$4.2E+2$	$3.31E-5$	0.013957	0.74058
10.0	0.047619	0.000495	$5.4E+2$	$2.36E-5$	0.012628	0.74192

(Largest tables in the world can be found in Potto Gas Tables at www.potto.org)

5.3.1 Isentropic Isothermal Flow Nozzle

General Relationship

In this section, the other extreme case model where the heat transfer to the gas is perfect, (e.g. Eckert number is very small) is presented. Again in reality the heat transfer is somewhere in between the two extremes. So, knowing the two limits provides a tool to examine where the reality should be expected. The perfect gas model is again assumed (later more complex models can be assumed and constructed in a future versions). In

isothermal process the perfect gas model reads

$$P = \rho\, R\, T \;\rightsquigarrow\; dP = d\rho\, R\, T \tag{5.75}$$

Substituting equation (5.75) into the momentum equation[6] yields

$$U\, dU + \frac{R\, T\, dP}{P} = 0 \tag{5.76}$$

Integration of equation (5.76) yields the Bernoulli's equation for ideal gas in isothermal process which reads

$$\rightsquigarrow \frac{{U_2}^2 - {U_1}^2}{2} + R\, T\, \ln \frac{P_2}{P_1} = 0 \tag{5.77}$$

Thus, the velocity at point 2 becomes

$$U_2 = \sqrt{2\, R\, T\, \ln \frac{P_2}{P_1} - {U_1}^2} \tag{5.78}$$

The velocity at point 2 for stagnation point, $U_1 \approx 0$ reads

$$U_2 = \sqrt{2\, R\, T\, \ln \frac{P_2}{P_1}} \tag{5.79}$$

Or in explicit terms of the stagnation properties the velocity is

$$U = \sqrt{2\, R\, T\, \ln \frac{P}{P_0}} \tag{5.80}$$

Transform from equation (5.77) to a dimensionless form becomes

$$\rightsquigarrow \frac{k\, R \overset{constant}{\cancel{T}} ({M_2}^2 - {M_1}^2)}{2} = R \overset{constant}{\cancel{T}} \ln \frac{P_2}{P_1} \tag{5.81}$$

Simplifying equation (5.81) yields

$$\rightsquigarrow \frac{k\, ({M_2}^2 - {M_1}^2)}{2} = \ln \frac{P_2}{P_1} \tag{5.82}$$

Or in terms of the pressure ratio equation (5.82) reads

Pressure Ratio

$$\frac{P_2}{P_1} = e^{\frac{k\, ({M_1}^2 - {M_2}^2)}{2}} = \left(\frac{e^{{M_1}^2}}{e^{{M_2}^2}} \right)^{\frac{k}{2}} \tag{5.83}$$

[6] The one dimensional momentum equation for steady state is $UdU/dx = -dP/dx + 0$(other effects) which are neglected here.

As oppose to the adiabatic case ($T_0 = constant$) in the isothermal flow the stagnation temperature ratio can be expressed

$$\frac{T_{01}}{T_{02}} = \frac{T_1}{T_2}\frac{\left(1+\frac{k-1}{2}{M_1}^2\right)}{\left(1+\frac{k-1}{2}{M_2}^2\right)} = \frac{\left(1+\frac{k-1}{2}{M_1}^2\right)}{\left(1+\frac{k-1}{2}{M_2}^2\right)} \tag{5.84}$$

T_0

Utilizing conservation of the mass $A\rho M = constant$ to yield

$$\frac{A_1}{A_2} = \frac{M_2 P_2}{M_1 P_1} \tag{5.85}$$

Combining equation (5.85) and equation (5.83) yields

Area Ratio

$$\frac{A_2}{A_1} = \frac{M_1}{M_2}\left(\frac{\mathrm{e}^{{M_2}^2}}{\mathrm{e}^{{M_1}^2}}\right)^{\frac{k}{2}} \tag{5.86}$$

The change in the stagnation pressure can be expressed as

Stagnation Pressure Ratio

$$\frac{P_{02}}{P_{01}} = \frac{P_2}{P_1}\left(\frac{1+\frac{k-1}{2}{M_2}^2}{1+\frac{k-1}{2}{M_1}^2}\right)^{\frac{k}{k-1}} = \left[\frac{\mathrm{e}^{{M_1}^2}}{\mathrm{e}^{{M_1}^2}}\right]^{\frac{k}{2}} \tag{5.87}$$

The critical point, at this stage, is unknown (at what Mach number the nozzle is choked is unknown) so there are two possibilities: the choking point or $M = 1$ to normalize the equation. Here the critical point defined as the point where $M = 1$ so the results can be compared to the adiabatic case and denoted by star. Again it has to emphasis that this critical point is not really related to physical critical point but it is arbitrary definition. The true critical point is when flow is choked and the relationship between two will be presented.

The critical pressure ratio can be obtained from (5.83) to read

Pressure Ratio

$$\frac{P}{P^*} = \frac{\rho}{\rho^*} = \mathrm{e}^{\frac{(1-M^2)\,k}{2}} \tag{5.88}$$

Equation (5.86) is reduced to obtained the critical area ratio writes

Area Ratio

$$\frac{A}{A^*} = \frac{1}{M}\mathrm{e}^{\frac{(1-M^2)\,k}{2}} \tag{5.89}$$

Similarly the stagnation temperature reads

Stagnation Temperature Ratio

$$\frac{T_0}{{T_0}^*} = \frac{2\left(1+\frac{k-1}{2}{M_1}^2\right)^{\frac{k}{k-1}}}{k+1} \tag{5.90}$$

Finally, the critical stagnation pressure reads

Stagnation Temperature Ratio

$$\frac{P_0}{{P_0}^*} = \mathrm{e}^{\frac{(1-M^2)\,k}{2}}\left(\frac{2\left(1+\frac{k-1}{2}{M_1}^2\right)}{k+1}\right)^{\frac{k}{k-1}} \tag{5.91}$$

The maximum value of stagnation pressure ratio is obtained when $M = 0$ at which is

$$\left.\frac{P_0}{{P_0}^*}\right|_{M=0} = \mathrm{e}^{\frac{k}{2}}\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \tag{5.92}$$

For specific heat ratio of $k = 1.4$, this maximum value is about two. It can be noted that the stagnation pressure is monotonically reduced during this process.

Of course in isothermal process $T = T^*$. All these equations are plotted in Figure (5.6). From the Figure 5.3 it can be observed that minimum of the curve A/A^* isn't on $M = 1$. The minimum of the curve is when area is minimum and at the point where the flow is choked. It should be noted that the stagnation temperature is not constant as in the adiabatic case and the critical point is the only one constant.

The mathematical procedure to find the minimum is simply taking the derivative and equating to zero as following

$$\frac{d\left(\frac{A}{A^*}\right)}{dM} = \frac{k\,M^2 e^{\frac{k\,(M^2-1)}{2}} - e^{\frac{k\,(M^2-1)}{2}}}{M^2} = 0 \tag{5.93}$$

Equation (5.93) simplified to

$$k\,M^2 - 1 = 0 \rightsquigarrow M = \frac{1}{\sqrt{k}} \tag{5.94}$$

It can be noticed that a similar results are obtained for adiabatic flow. The velocity at the throat of isothermal model is smaller by a factor of $\sqrt{k}$. Thus, dividing the critical adiabatic velocity by $\sqrt{k}$ results in

$$U_{throat_{max}} = \sqrt{RT} \tag{5.95}$$

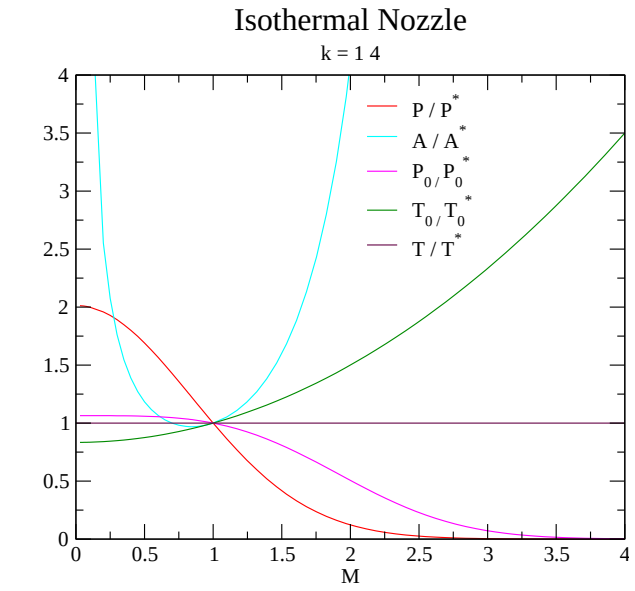


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Fig. -5.6. Various ratios as a function of Mach number for isothermal Nozzle

On the other hand, the pressure loss in adiabatic flow is milder as can be seen in Figure (5.7(a)). The heat transfer compensated the pressure loss.

It should be emphasized that the stagnation pressure decrees. It is convenient to find expression for the ratio of the initial stagnation pressure (the stagnation pressure before entering the nozzle) to the pressure at the throat. Utilizing equation (5.88) the following relationship can be obtained

$$\frac{P_{throat}}{P_{0_{initial}}} = \frac{P^*}{P_{0_{initial}}} \frac{P_{throat}}{P^*} =$$

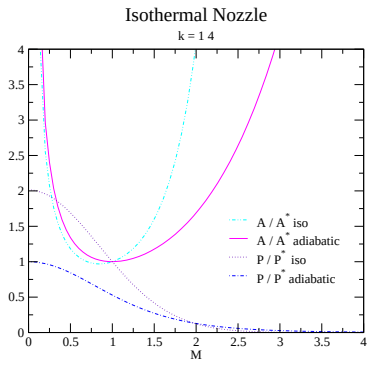
$$\frac{1}{\mathsf{e}^{\frac{(1-0^2)k}{2}}} \mathsf{e}^{\left(1-\left(\frac{1}{\sqrt{k}}\right)^2\right)\frac{k}{2}} =$$

$$\mathsf{e}^{-\frac{1}{2}} = 0.60653 \tag{5.96}$$

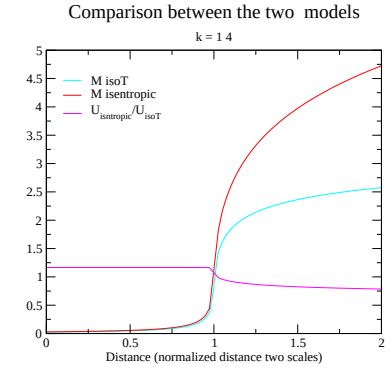
Notice that the critical pressure is independent of the specific heat ratio, k, as opposed to the adiabatic case. It also has to be emphasized that the stagnation values of the isothermal model are not constant. Again, the heat transfer is expressed as

$$Q = C_p \left(T_{0_2} - T_{0_2}\right) \tag{5.97}$$

For comparison between the adiabatic model and the isothermal a simple profile of



(a) Comparison between the isothermal nozzle and adiabatic nozzle in various variables



(b) The comparison of the adiabatic model and isothermal model

Fig. -5.7. The comparison of nozzle flow

nozzle area as a function of the distance is assumed. This profile isn't an ideal profile but rather a simple sample just to examine the difference between the two models so in an actual situation it can be bounded. To make sense and eliminate unnecessary details the distance from the entrance to the throat is normalized (to one (1)). In the same fashion the distance from the throat to the exit is normalized (to one (1)) (it doesn't mean that these distances are the same). In this comparison the entrance area ratio and the exit area ratio are the same and equal to 20. The Mach number was computed for the two models and plotted in Figure (5.7(b)). In this comparison it has to be remembered that critical area for the two models are different by about 3% (for $k = 1.4$). As can be observed from Figure (5.7(b)). The Mach number for the isentropic is larger for the supersonic branch but the velocity is lower. The ratio of the velocities can be expressed as

$$\frac{U_s}{U_T} = \frac{M_s\sqrt{k\,R\,T_s}}{M_T\sqrt{k\,R\,T_s}} \tag{5.98}$$

It can be noticed that temperature in the isothermal model is constant while temperature in the adiabatic model can be expressed as a function of the stagnation temperature. The initial stagnation temperatures are almost the same and can be canceled out to obtain

$$\frac{U_s}{U_T} \sim \frac{M_s}{M_T\sqrt{1+\frac{k-1}{2}{M_s}^2}} \tag{5.99}$$

By utilizing equation (5.99), the velocity ratio was obtained and is plotted in Figure 5.7(b).

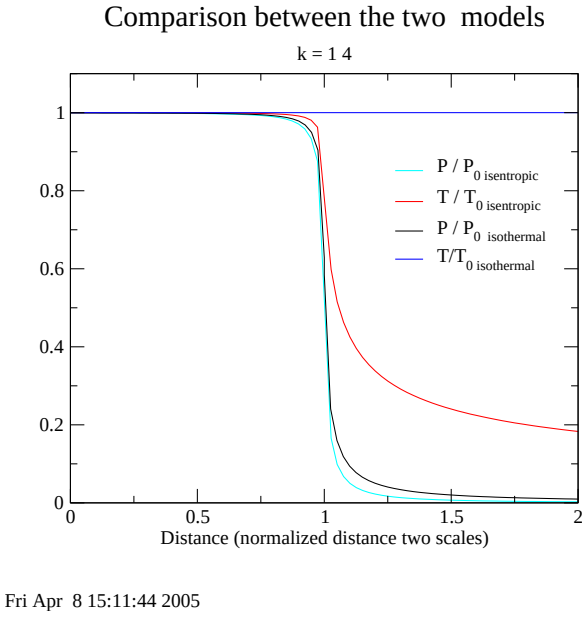


Fig. -5.8. Comparison of the pressure and temperature drop as a function of the normalized length (two scales)

Thus, using the isentropic model results in under prediction of the actual results for the velocity in the supersonic branch. While, the isentropic for the subsonic branch will be over prediction. The prediction of the Mach number are similarly shown in Figure 5.7(b).

Two other ratios need to be examined: temperature and pressure. The initial stagnation temperature is denoted as $T_{0_{int}}$. The temperature ratio of $T/T_{0_{int}}$ can be obtained via the isentropic model as

$$\frac{T}{T_{0_{int}}} = \frac{1}{1 + \frac{k-1}{2} M^2} \tag{5.100}$$

While the temperature ratio of the isothermal model is constant and equal to one (1). The pressure ratio for the isentropic model is

$$\frac{P}{P_{0_{int}}} = \frac{1}{\left(1 + \frac{k-1}{2} M^2\right)^{\frac{k-1}{k}}} \tag{5.101}$$

and for the isothermal process the stagnation pressure varies and has to be taken into

account as the following:

$$\frac{P_z}{P_{0_{int}}} = \frac{{P_0}^*}{P_{0_{int}}} \frac{P_{0_z}}{{P_0}^*} \overbrace{\frac{P_z}{P_{0_z}}}^{isentropic} \tag{5.102}$$

where z is an arbitrary point on the nozzle. Using equations (5.87) and the isentropic relationship, the sought ratio is provided.

Figure (5.8) shows that the range between the predicted temperatures of the two models is very large, while the range between the predicted pressure by the two models is relatively small. The meaning of this analysis is that transferred heat affects the temperature to a larger degree but the effect on the pressure is much less significant.

To demonstrate the relativity of the approach advocated in this book consider the following example.

Example 5.10:
Consider a diverging–converging nozzle made out of wood (low conductive material) with exit area equal entrance area. The throat area ratio to entrance area is 1:4 respectively. The stagnation pressure is 5[Bar] *and the stagnation temperature is* $27^{\circ}\mathrm{C}$. *Assume that the back pressure is low enough to have supersonic flow without shock and* $k = 1.4$. *Calculate the velocity at the exit using the adiabatic model. If the nozzle was made from copper (a good heat conductor) a larger heat transfer occurs, should the velocity increase or decrease? What is the maximum possible increase?*

SOLUTION

The first part of the question deals with the adiabatic model i.e. the conservation of the stagnation properties. Thus, with known area ratio and known stagnation Potto–GDC provides the following table:

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
0.14655	0.99572	0.98934	4.0000	0.98511	3.9405
2.9402	0.36644	0.08129	4.0000	0.02979	0.11915

With the known Mach number and temperature at the exit, the velocity can be calculated. The exit temperature is $0.36644 \times 300 = 109.9K$. The exit velocity, then, is

$$U = M\sqrt{k\,R\,T} = 2.9402\sqrt{1.4 \times 287 \times 109.9} \sim 617.93[m/sec]$$

Even for the isothermal model, the initial stagnation temperature is given as $300K$. Using the area ratio in Figure (5.6) or using the Potto–GDC obtains the following table

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
1.9910	1.4940	0.51183	4.0000	0.12556	0.50225

The exit Mach number is known and the initial temperature to the throat temperature ratio can be calculated as the following:

$$\frac{T_{0_{ini}}}{{T_0}^*} = \frac{1}{1+\frac{k-1}{2}\frac{1}{k}} = \frac{1}{1+\frac{k-1}{k}} = 0.777777778$$

Thus the stagnation temperature at the exit is

$$\frac{T_{0_{ini}}}{T_{0_{exit}}} = 1.4940/0.777777778 = 1.921$$

The exit stagnation temperature is $1.92 \times 300 = 576.2K$. The exit velocity can be determined by utilizing the following equation

$$U_{exit} = M\sqrt{k\,R\,T} = 1.9910\sqrt{1.4 \times 287 \times 300.0} = 691.253[m/sec]$$

As was discussed before, the velocity in the copper nozzle will be larger than the velocity in the wood nozzle. However, the maximum velocity cannot exceed the $691.253[m/sec]$

End Solution

5.4 The Impulse Function

5.4.1 Impulse in Isentropic Adiabatic Nozzle

One of the functions that is used in calculating the forces is the Impulse function. The Impulse function is denoted here as F, but in the literature some denote this function as I. To explain the motivation for using this definition consider the calculation of the net forces that acting on section shown in Figure (5.9). To calculate the net forces acting in the x–direction the momentum equation has to be applied

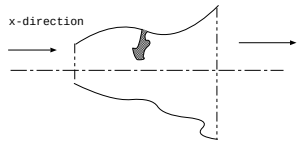


Fig. -5.9. Schematic to explain the significances of the Impulse function.

$$F_{net} = \dot{m}(U_2 - U_1) + P_2A_2 - P_1A_1 \tag{5.103}$$

The net force is denoted here as F_{net}. The mass conservation also can be applied to our control volume

$$\dot{m} = \rho_1 A_1 U_1 = \rho_2 A_2 U_2 \tag{5.104}$$

Combining equation (5.103) with equation (5.104) and by utilizing the identity in equation (5.41) results in

$$F_{net} = kP_2A_2{M_2}^2 - kP_1A_1{M_1}^2 + P_2A_2 - P_1A_1 \tag{5.105}$$

Rearranging equation (5.105) and dividing it by P_0A^* results in

$$\frac{F_{net}}{P_0A^*} = \overbrace{\frac{P_2A_2}{P_0A^*}}^{f(M_2)}\overbrace{\left(1+kM_2{}^2\right)}^{f(M_2)} - \overbrace{\frac{P_1A_1}{P_0A^*}}^{f(M_1)}\overbrace{\left(1+kM_1{}^2\right)}^{f(M_1)} \tag{5.106}$$

Examining equation (5.106) shows that the right hand side is only a function of Mach number and specific heat ratio, k. Hence, if the right hand side is only a function of the Mach number and k than the left hand side must be function of only the same parameters, M and k. Defining a function that depends only on the Mach number creates the convenience for calculating the net forces acting on any device. Thus, defining the Impulse function as

$$F = PA\left(1+kM_2{}^2\right) \tag{5.107}$$

In the Impulse function when F $(M=1)$ is denoted as F^*

$$F^* = P^*A^*\left(1+k\right) \tag{5.108}$$

The ratio of the Impulse function is defined as

$$\frac{F}{F^*} = \frac{P_1A_1}{P^*A^*}\frac{\left(1+kM_1{}^2\right)}{(1+k)} = \frac{1}{\underbrace{\frac{P^*}{P_0}}_{\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}}} \overbrace{\frac{P_1A_1}{P_0A^*}\left(1+kM_1{}^2\right)}^{\text{see function (5.106)}} \frac{1}{(1+k)} \tag{5.109}$$

This ratio is different only in a coefficient from the ratio defined in equation (5.106) which makes the ratio a function of k and the Mach number. Hence, the net force is

$$F_{net} = P_0A^*(1+k)\left(\frac{k+1}{2}\right)^{\frac{k}{k-1}}\left(\frac{F_2}{F^*} - \frac{F_1}{F^*}\right) \tag{5.110}$$

To demonstrate the usefulness of the this function consider a simple situation of the flow through a converging nozzle.

Example 5.11:

Consider a flow of gas into a converging nozzle with a mass flow rate of $1[kg/sec]$ *and the entrance area is* $0.009[m^2]$ *and the exit area is* $0.003[m^2]$. *The stagnation temperature is* $400K$ *and the pressure at point 2 was measured as* $5[Bar]$. *Calculate the net force acting on the nozzle and pressure at point 1.*

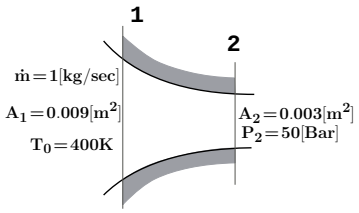


Fig. -5.10. Schematic of a flow of a compressible substance (gas) through a converging nozzle for example (5.11)

SOLUTION

The solution is obtained by getting the data for the Mach number. To obtained the Mach number, the ratio of P_1A_1/A^*P_0 is needed to be calculated. To obtain this ratio the denominator is needed to be obtained. Utilizing Fliegner's equation (5.50), provides the following

$$A^*P_0 = \frac{\dot{m}\sqrt{RT}}{0.058} = \frac{1.0 \times \sqrt{400 \times 287}}{0.058} \sim 70061.76[N]$$

and

$$\frac{A_2P_2}{A^*P_0} = \frac{500000 \times 0.003}{70061.76} \sim 2.1$$

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.27353	0.98526	0.96355	2.2121	0.94934	2.1000	0.96666

With the area ratio of $\frac{A}{A^\star} = 2.2121$ the area ratio of at point 1 can be calculated.

$$\frac{A_1}{A^\star} = \frac{A_2}{A^\star}\frac{A_1}{A_2} = 2.2121 \times \frac{0.009}{0.003} = 5.2227$$

And utilizing again Potto-GDC provides

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.11164	0.99751	0.99380	5.2227	0.99132	5.1774	2.1949

The pressure at point 1 is

$$P_1 = P_2\frac{P_0}{P_2}\frac{P_1}{P_0} = 5.0 times 0.94934/0.99380 \sim 4.776[Bar]$$

The net force is obtained by utilizing equation (5.110)

$$F_{net} = P_2 A_2 \frac{P_0 A^*}{P_2 A_2}(1+k)\left(\frac{k+1}{2}\right)^{\frac{k}{k-1}}\left(\frac{F_2}{F^*} - \frac{F_1}{F^*}\right)$$
$$= 500000 \times \frac{1}{2.1} \times 2.4 \times 1.2^{3.5} \times (2.1949 - 0.96666) \sim 614[kN]$$

End Solution

5.4.2 The Impulse Function in Isothermal Nozzle

Previously Impulse function was developed in the isentropic adiabatic flow. The same is done here for the isothermal nozzle flow model. As previously, the definition of the Impulse function is reused. The ratio of the impulse function for two points on the nozzle is

$$\frac{F_2}{F_1} = \frac{P_2 A_2 + \rho_2 {U_2}^2 A_2}{P_1 A_1 + \rho_1 {U_1}^2 A_1} \tag{5.111}$$

Utilizing the ideal gas model for density and some rearrangement results in

$$\frac{F_2}{F_1} = \frac{P_2 A_2}{P_1 A_1}\frac{1 + \frac{{U_2}^2}{RT}}{1 + \frac{{U_1}^2}{RT}} \tag{5.112}$$

Since $U^2/RT = k\,M^2$ and the ratio of equation (5.85) transformed equation into (5.112)

$$\frac{F_2}{F_1} = \frac{M_1}{M_2}\frac{1 + k{M_2}^2}{1 + k{M_1}^2} \tag{5.113}$$

At the star condition ($M = 1$) (not the minimum point) results in

Impulse Ratio

$$\frac{F_2}{F^*} = \frac{1}{M_2}\frac{1 + k\,{M_2}^2}{1 + k} \tag{5.114}$$

5.5 Isothermal Table

Table -5.3. Isothermal Table

$\mathbf{M}$	$\frac{\mathbf{T_0}}{\mathbf{T_0^\star}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0^\star}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P^\star}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.00	0.52828	1.064	$5.0E+5$	2.014	$1.0E$+6	$4.2E$+5
0.05	0.52921	1.064	9.949	2.010	20.00	8.362

Table -5.3. Isothermal Table (continue)

M	$\frac{T_0}{T_0^\star}$	$\frac{P_0}{P_0^\star}$	$\frac{A}{A^\star}$	$\frac{P}{P^\star}$	$\frac{A \times P}{A^* \times P_0}$	$\frac{F}{F^*}$
0.1	0.53199	1.064	5.001	2.000	10.00	4.225
0.2	0.54322	1.064	2.553	1.958	5.000	2.200
0.3	0.56232	1.063	1.763	1.891	3.333	1.564
0.4	0.58985	1.062	1.389	1.800	2.500	1.275
0.5	0.62665	1.059	1.183	1.690	2.000	1.125
0.6	0.67383	1.055	1.065	1.565	1.667	1.044
0.7	0.73278	1.047	0.99967	1.429	1.429	1.004
0.8	0.80528	1.036	0.97156	1.287	1.250	0.98750
0.9	0.89348	1.021	0.97274	1.142	1.111	0.98796
1.0	1.000	1.000	1.000	1.000	1.000	1.000
1.1	1.128	0.97376	1.053	0.86329	0.90909	1.020
1.2	1.281	0.94147	1.134	0.73492	0.83333	1.047
1.3	1.464	0.90302	1.247	0.61693	0.76923	1.079
1.4	1.681	0.85853	1.399	0.51069	0.71429	1.114
1.5	1.939	0.80844	1.599	0.41686	0.66667	1.153
1.6	2.245	0.75344	1.863	0.33554	0.62500	1.194
1.7	2.608	0.69449	2.209	0.26634	0.58824	1.237
1.8	3.035	0.63276	2.665	0.20846	0.55556	1.281
1.9	3.540	0.56954	3.271	0.16090	0.52632	1.328
2.0	4.134	0.50618	4.083	0.12246	0.50000	1.375
2.5	9.026	0.22881	15.78	0.025349	0.40000	1.625
3.0	19.41	0.071758	90.14	0.00370	0.33333	1.889
3.5	40.29	0.015317	$7.5E+2$	0.000380	0.28571	2.161
4.0	80.21	0.00221	$9.1E+3$	$2.75E{-}5$	0.25000	2.438
4.5	$1.5E+2$	0.000215	$1.6E+5$	$1.41E{-}6$	0.22222	2.718

Table -5.3. Isothermal Table (continue)

$\mathbf{M}$	$\frac{\mathbf{T_0}}{\mathbf{T_0{}^\star}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0{}^\star}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P^\star}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
5.0	$2.8E+2$	$1.41E{-}5$	$4.0E+6$	0.0	0.20000	3.000
5.5	$4.9E+2$	0.0	$1.4E+8$	0.0	0.18182	3.284
6.0	$8.3E+2$	0.0	$7.3E+9$	0.0	0.16667	3.569
6.5	$1.4E+3$	0.0	$5.3E{+}11$	0.0	0.15385	3.856
7.0	$2.2E+3$	0.0	$5.6E{+}13$	0.0	0.14286	4.143
7.5	$3.4E+3$	0.0	$8.3E{+}15$	0.0	0.13333	4.431
8.0	$5.2E+3$	0.0	$1.8E{+}18$	0.0	0.12500	4.719
8.5	$7.7E+3$	0.0	$5.4E{+}20$	0.0	0.11765	5.007
9.0	$1.1E+4$	0.0	$2.3E{+}23$	0.0	0.11111	5.296
9.5	$1.6E+4$	0.0	$1.4E{+}26$	0.0	0.10526	5.586
10.	$2.2E+4$	0.0	$1.2E{+}29$	0.0	0.100000	5.875

5.6 The effects of Real Gases

To obtained expressions for non–ideal gas it is commonly done by reusing the ideal gas model and introducing a new variable which is a function of the gas properties like the critical pressure and critical temperature. Thus, a real gas equation can be expressed in equation (4.22). Differentiating equation (4.22) and dividing by equation (4.22) yields

$$\frac{dP}{P} = \frac{dz}{z} + \frac{d\rho}{\rho} + \frac{dT}{T} \tag{5.115}$$

Again, Gibb's equation (5.26) is reused to related the entropy change to the change in thermodynamics properties and applied on non-ideal gas. Since $ds = 0$ and utilizing the equation of the state $dh = dP/\rho$. The enthalpy is a function of the temperature and pressure thus, $h = h(T, P)$ and full differential is

$$dh = \left(\frac{\partial h}{\partial T}\right)_P dT + \left(\frac{\partial h}{\partial P}\right)_T dP \tag{5.116}$$

The definition of pressure specific heat is $C_p \equiv \frac{\partial h}{\partial T}$ and second derivative is Maxwell relation hence,

$$\left(\frac{\partial h}{\partial P}\right)_T = v - T\left(\frac{\partial s}{\partial T}\right)_P \tag{5.117}$$

First, the differential of enthalpy is calculated for real gas equation of state as

$$dh = C_p dT - \left(\frac{T}{Z}\right)\left(\frac{\partial z}{\partial T}\right)_P \frac{dP}{\rho} \tag{5.118}$$

Equations (5.26) and (4.22) are combined to form

$$\frac{ds}{R} = \frac{C_p}{R}\frac{dT}{T} - z\left[1 + \left(\frac{T}{Z}\right)\left(\frac{\partial z}{\partial T}\right)_P\right]\frac{dP}{P} \tag{5.119}$$

The mechanical energy equation can be expressed as

$$\int d\left(\frac{U^2}{2}\right) = -\int \frac{dP}{\rho} \tag{5.120}$$

At the stagnation the definition requires that the velocity is zero. To carry the integration of the right hand side the relationship between the pressure and the density has to be defined. The following power relationship is assumed

$$\frac{\rho}{\rho_0} = \left(\frac{P}{P_0}\right)^{\frac{1}{n}} \tag{5.121}$$

Notice, that for perfect gas the n is substituted by k. With integration of equation (5.120) when using relationship which is defined in equation (5.121) results

$$\frac{U^2}{2} = \int_{P_0}^{P_1} \frac{dP}{\rho} = \int_{P_0}^{P} \frac{1}{\rho_0}\left(\frac{P_0}{P}\right)^{\frac{1}{n}} dP \tag{5.122}$$

Substituting relation for stagnation density (4.22) results

$$\frac{U^2}{2} = \int_{P_0}^{P} \frac{z_0 R T_0}{P_0}\left(\frac{P_0}{P}\right)^{\frac{1}{n}} dP \tag{5.123}$$

For $n > 1$ the integration results in

$$U = \sqrt{z_0 R T_0 \frac{2n}{n-1}\left[1 - \left(\frac{P}{P_0}\right)^{\left(\frac{n-1}{n}\right)}\right]} \tag{5.124}$$

For $n = 1$ the integration becomes

$$U = \sqrt{2 z_0 R T_0 \ln\left(\frac{P_0}{P}\right)} \tag{5.125}$$

It must be noted that n is a function of the critical temperature and critical pressure. The mass flow rate is regardless to equation of state as following

$$\dot{m} = \rho^* A^* U^* \tag{5.126}$$

Where ρ^* is the density at the throat (assuming the chocking condition) and A^* is the cross area of the throat. Thus, the mass flow rate in our properties

$$\dot{m} = A^* \overbrace{\frac{P_0}{z_0 R T_0}\left(\frac{P}{P_0}\right)^{\frac{1}{n}}}^{\rho^*} \overbrace{\sqrt{z_0 R T_0 \frac{2n}{n-1}\left[1-\left(\frac{P}{P_0}\right)^{\left(\frac{n-1}{n}\right)}\right]}}^{U^*} \tag{5.127}$$

For the case of $n = 1$

$$\dot{m} = A^* \overbrace{\frac{P_0}{z_0 R T_0}\left(\frac{P}{P_0}\right)^{\frac{1}{n}}}^{\rho^*} \overbrace{\sqrt{2 z_0 R T_0 \ln\left(\frac{P_0}{P}\right)}}^{U**} \tag{5.128}$$

The Mach number can be obtained by utilizing equation (4.37) to defined the Mach number as

$$M = \frac{U}{\sqrt{znRT}} \tag{5.129}$$

Integrating equation (5.119) when $ds = 0$ results

$$\int_{T_1}^{T_2} \frac{C_p}{R}\frac{dT}{T} = \int_{P_1}^{P_2} z\left(1 + \left(\frac{T}{Z}\right)\left(\frac{\partial z}{\partial T}\right)_P \frac{dP}{P}\right) \tag{5.130}$$

To carryout the integration of equation (5.130) looks at Bernnolli's equation which is

$$\int \frac{dU^2}{2} = -\int \frac{dP}{\rho} \tag{5.131}$$

After integration of the velocity

$$\frac{dU^2}{2} = -\int_1^{P/P_0} \frac{\rho_0}{\rho} d\left(\frac{P}{P_0}\right) \tag{5.132}$$

It was shown in Chapter (4) that (4.36) is applicable for some ranges of relative temperature and pressure (relative to critical temperature and pressure and not the stagnation conditions).

$$U = \sqrt{z_0 R T_0 \left(\frac{2n}{n-1}\right)\left[1-\left(\frac{P}{P_0}\right)^{\frac{n-1}{n}}\right]} \tag{5.133}$$

When $n = 1$ or when $n \to 1$

$$U = \sqrt{2 z_0 R T_0 \ln\left(\frac{P_0}{P}\right)} \tag{5.134}$$

The mass flow rate for the real gas $\dot{m} = \rho^* U^* A^*$

$$\dot{m} = \frac{A^* P_0}{\sqrt{z_0 R T_0}} \sqrt{\frac{2n}{n-1}} \left(\frac{P^*}{P_0}\right)^{\frac{1}{n}} \left[1 - \frac{P^*}{P_0}\right] \tag{5.135}$$

And for $n = 1$

$$\dot{m} = \frac{A^* P_0}{\sqrt{z_0 R T_0}} \sqrt{\frac{2n}{n-1}} \sqrt{2 z_0 R T_0 \ln\left(\frac{P_0}{P}\right)} \tag{5.136}$$

Fliegner's number in this case is

$$Fn = \frac{\dot{m} c_0}{A^* P_0} \sqrt{\frac{2n}{n-1}} \left(\frac{P^*}{P_0}\right)^{\frac{1}{n}} \left[1 - \frac{P^*}{P_0}\right] \tag{5.137}$$

Fliegner's number for $n = 1$ is

$$Fn = \frac{\dot{m} c_0}{A^* P_0} = 2 \left(\frac{P^*}{P_0}\right)^2 - \ln\left(\frac{P^*}{P_0}\right) \tag{5.138}$$

The critical ratio of the pressure is

$$\frac{P^*}{P_0} = \left(\frac{2}{n+1}\right)^{\frac{n}{n-1}} \tag{5.139}$$

When $n = 1$ or more generally when $n \to 1$ this is a ratio approach

$$\frac{P^*}{P_0} = \sqrt{\mathrm{e}} \tag{5.140}$$

To obtain the relationship between the temperature and pressure, equation (5.130) can be integrated

$$\frac{T_0}{T} = \left(\frac{P_0}{P}\right)^{\frac{R}{C_p}\left[z + T\left(\frac{\partial z}{\partial T}\right)_P\right]} \tag{5.141}$$

The power of the pressure ratio is approaching $\frac{k-1}{k}$ when z approaches 1. Note that

$$\frac{T_0}{T} = \left(\frac{z_0}{z}\right) \left(\frac{P_0}{P}\right)^{\frac{1-n}{n}} \tag{5.142}$$

The Mach number at every point at the nozzle can be expressed as

$$M = \sqrt{\left(\frac{2}{n-1}\right) \frac{z_0}{z} \frac{T_0}{T} \left[1 - \left(\frac{P-0}{P}\right)^{\frac{1-n}{n}}\right]} \tag{5.143}$$

For $n = 1$ the Mach number is

$$M = \sqrt{2\frac{z_0}{z}\frac{T_0}{T}\ln\frac{P_0}{P}} \tag{5.144}$$

The pressure ratio at any point can be expressed as a function of the Mach number as

$$\frac{T_0}{T} = \left[1 + \frac{n-1}{2}M^2\right]^{\left(\frac{n-1}{n}\right)\left[z+T\left(\frac{\partial z}{\partial T}\right)_P\right]} \tag{5.145}$$

for $n = 1$

$$\frac{T_0}{T} = \mathsf{e}^{M^2\left[z+T\left(\frac{\partial z}{\partial T}\right)_P\right]} \tag{5.146}$$

The critical temperature is given by

$$\frac{T^*}{T_0} = \left(\frac{1+n}{2}\right)^{\left(\frac{n}{1-n}\right)\left[z+T\left(\frac{\partial z}{\partial T}\right)_P\right]} \tag{5.147}$$

and for $n = 1$

$$\frac{T^*}{T_0} = \sqrt{\mathsf{e}^{-\left[z+T\left(\frac{\partial z}{\partial T}\right)_P\right]}} \tag{5.148}$$

The mass flow rate as a function of the Mach number is

$$\dot{m} = \frac{P_0 n}{c_0} M \sqrt{\left(1 + \frac{n-1}{2}M^2\right)^{\frac{n+1}{n-1}}} \tag{5.149}$$

For the case of $n = 1$ the mass flow rate is

$$\dot{m} = \frac{P_0 A^* n}{c_0}\sqrt{\mathsf{e}^{M^2}}\sqrt{\left(1 + \frac{n-1}{2}M^2\right)^{\frac{n+1}{n-1}}} \tag{5.150}$$

Example 5.12:
A design is required that at a specific point the Mach number should be $M = 2.61$, the pressure $2[Bar]$, and temperature $300K$.

i. Calculate the area ratio between the point and the throat.

ii. Calculate the stagnation pressure and the stagnation temperature.

iii. Are the stagnation pressure and temperature at the entrance different from the point? You can assume that $k = 1.405$.

SOLUTION

1. The solution is simplified by using Potto-GDC for $M = 2.61$ the results are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
2.6100	0.42027	0.11761	2.9066	0.04943	0.14366

2. The stagnation pressure is obtained from

$$P_0 = \frac{P_0}{P} P = \frac{2.61}{0.04943} \sim 52.802[Bar]$$

   The stagnation temperature is

$$T_0 = \frac{T_0}{T} T = \frac{300}{0.42027} \sim 713.82K$$

3. Of course, the stagnation pressure is constant for isentropic flow.

End Solution

5.7 Isentropic Relationships for Real Gases

Recently there was a push to utilize the real gas model. In this section a brief discussion on this point is presented. The development of that leads to equation (5.9) has to be modified. In this derivation it is assumed the heat capacities to be constant as well in the ideal gas model. Hence equation (5.2) as well equation (5.5) are valid. Hence the staring point is

$$1 + \frac{U^2}{2\,C_p T} = \frac{T_0}{T} \tag{5.151}$$

Substituting into equation (5.151) for $c^2 = n\,Z\,R\,T$ (equation (4.37)) yields

$$\frac{T_0}{T} = 1 + \frac{n\,Z\,R\,U^2}{2\,C_p\,c^2} \tag{5.152}$$

By utilizing the definition of k by equation (2.24) and inserting it into equation (5.6) yields

$$\frac{T_0}{T} = 1 + \frac{Z\,n\,(k-1)}{2\,k}\,\frac{U^2}{c^2} \tag{5.153}$$

Or in a dimensionless form as

$$\frac{T_0}{T} = 1 + \frac{Z\,n\,(k-1)}{2\,k}\,M^2 \tag{5.154}$$

Since n/k is expressed by equation (4.35) resulting in

$$\frac{T_0}{T} = 1 + \frac{Z\,(k-1)}{2}\left(\frac{Z + T\left(\dfrac{\partial Z}{\partial T}\right)_\rho}{Z + T\left(\dfrac{\partial z}{\partial T}\right)_P}\right) M^2 \tag{5.155}$$

The pressure relationship is obtained[7] from equation (4.36) as

$$\frac{P_1}{P_0} = \left(\frac{\rho_1}{\rho_0}\right)^n \tag{5.156}$$

Inserting equation (2.39) into equation (5.156) to provide

$$\left(\frac{\rho_1}{\rho_0}\right)^n = \frac{Z_1\,\rho_1 \cancel{R} T_1}{Z_0\,\rho_0 \cancel{R} T_0} \qquad \Longrightarrow \qquad \frac{T_0}{T_1} = \frac{Z_1}{Z_0}\left(\frac{\rho_0}{\rho_1}\right)^{1-n} \tag{5.157}$$

Using the identy of $\rho = P\,/zT\,R$ and substituting into equation (5.156) to provide

$$\begin{aligned} \frac{P_1}{P_0} = \left(\frac{\dfrac{P_1}{Z_1\,T_1\cancel{R}}}{\dfrac{P_0}{Z_0\,T_0\cancel{R}}}\right)^n &= \left(\frac{P_1}{P_0}\right)^n \left(\frac{Z_0\,T_0}{Z_1\,T_1}\right)^n \\ \frac{T_0}{T_1} &= \frac{Z_1}{Z_0}\left(\frac{P_1}{P_0}\right)^{\frac{1-n}{n}} \end{aligned} \tag{5.158}$$

Hence the relatinshp for the static and stagantion pressure (equation (5.155)) are

$$\frac{P_1}{P_0} = \left(\frac{Z_0}{Z_1}\left[1 + \frac{Z\,(k-1)}{2}\left(\frac{Z + T\left(\dfrac{\partial Z}{\partial T}\right)_\rho}{Z + T\left(\dfrac{\partial z}{\partial T}\right)_P}\right) M^2\right]\right)^{\frac{n}{1-n}} \tag{5.159}$$

[7] Some of corrections where suggested by Guy de Carufel

CHAPTER 6

Normal Shock

In this chapter the relationships between the two sides of normal shock are presented. In this discussion, the flow is assumed to be in a steady state, and the thickness of the shock is assumed to be very small. A discussion on the shock thickness will be presented in a forthcoming section[1].

A shock can occur in at least two different mechanisms. The first is when a large difference (above a small minimum value) between the two sides of a membrane, and when the membrane bursts (see the discussion about the shock tube). Of course, the shock travels from the high pressure to the low pressure side. The second is when many sound waves "run into" each other and accumulate (some refer to it as "coalescing") into a large difference, which is the shock wave. In fact, the sound wave can be viewed as an extremely weak shock. In the speed of sound analysis, it was assumed the medium is continuous, without any abrupt changes. This assumption is no longer valid in the case of a shock. Here, the relationship for a perfect gas is constructed.

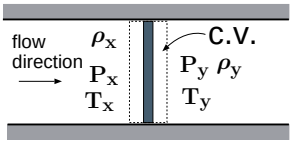


Fig. -6.1. A shock wave inside a tube, but it can also be viewed as a one–dimensional shock wave.

In Figure 6.1 a control volume for this analysis is shown, and the gas flows from left to right. The conditions, to the left and to the right of the shock, are assumed to be uniform[2]. The conditions to the right of the shock wave are uniform, but different from the left side. The transition in the shock is abrupt and in a very narrow width.

The chemical reactions (even condensation) are neglected, and the shock occurs

[1] Currently under construction.

[2] Clearly the change in the shock is so significant compared to the changes in medium before and after the shock that the changes in the mediums (flow) can be considered uniform.

at a very narrow section. Clearly, the isentropic transition assumption is not appropriate in this case because the shock wave is a discontinued area. Therefore, the increase of the entropy is fundamental to the phenomenon and the understanding of it.

It is further assumed that there is no friction or heat loss at the shock (because the heat transfer is negligible due to the fact that it occurs on a relatively small surface). It is customary in this field to denote x as the upstream condition and y as the downstream condition.

The mass flow rate is constant from the two sides of the shock and therefore the mass balance is reduced to

$$\rho_x U_x = \rho_y U_y \tag{6.1}$$

In a shock wave, the momentum is the quantity that remains constant because there are no external forces. Thus, it can be written that

$$P_x - P_y = \left(\rho_x {U_y}^2 - \rho_y {U_x}^2\right) \tag{6.2}$$

The process is adiabatic, or nearly adiabatic, and therefore the energy equation can be written as

$$C_p\, T_x + \frac{{U_x}^2}{2} = C_p\, T_y + \frac{{U_y}^2}{2} \tag{6.3}$$

The equation of state for perfect gas reads

$$P = \rho\, R T \tag{6.4}$$

If the conditions upstream are known, then there are four unknown conditions downstream. A system of four unknowns and four equations is solvable. Nevertheless, one can note that there are two solutions because of the quadratic of equation (6.3). These two possible solutions refer to the direction of the flow. Physics dictates that there is only one possible solution. One cannot deduce the direction of the flow from the pressure on both sides of the shock wave. The only tool that brings us to the direction of the flow is the second law of thermodynamics. This law dictates the direction of the flow, and as it will be shown, the gas flows from a supersonic flow to a subsonic flow. Mathematically, the second law is expressed by the entropy. For the adiabatic process, the entropy must increase. In mathematical terms, it can be written as follows:

$$s_y - s_x > 0 \tag{6.5}$$

Note that the greater–equal signs were not used. The reason is that the process is irreversible, and therefore no equality can exist. Mathematically, the parameters are P, T, U, and ρ, which are needed to be solved. For ideal gas, equation (6.5) is

$$\ln\left(\frac{T_y}{T_x}\right) - (k-1)\,\frac{P_y}{P_x} > 0 \tag{6.6}$$

It can also be noticed that entropy, s, can be expressed as a function of the other parameters. Now one can view these equations as two different subsets of equations. The first set is the energy, continuity, and state equations, and the second set is the momentum, continuity, and state equations. The solution of every set of these equations produces one additional degree of freedom, which will produce a range of possible solutions. Thus, one can have a whole range of solutions. In the first case, the energy equation is used, producing various resistance to the flow. This case is called Fanno flow, and Chapter 10 deals extensively with this topic. The mathematical explanation is given Chapter (10) in greater detail. Instead of solving all the equations that were presented, one can solve only four (4) equations (including the second law), which will require additional parameters. If the energy, continuity, and state equations are solved for the arbitrary value of the T_y, a parabola in the $T--s$ diagram will be obtained. On the other hand, when the momentum equation is solved instead of the energy equation, the degree of freedom is now energy, i.e., the energy amount "added" to the shock. This situation is similar to a frictionless flow with the addition of heat, and this flow is known as Rayleigh flow. This flow is dealt with in greater detail in Chapter (11).

Since the shock has no heat transfer (a special case of Rayleigh flow) and there isn't essentially any momentum transfer (a special case of Fanno flow), the intersection of these two curves is what really happened in the shock. In Figure 6.2, the intersection is shown and two solutions are obtained. Clearly, the increase of the entropy determines the direction of the flow. The entropy increases from point x to point y. It is also worth noting that the temperature at $M = 1$ on Rayleigh flow is larger than that on the Fanno line.

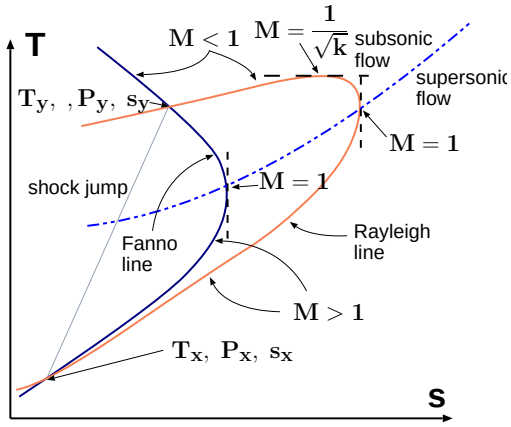


Fig. -6.2. The intersection of Fanno flow and Rayleigh flow produces two solutions for the shock wave.

6.1 Solution of the Governing Equations

6.1.1 Informal Model

Accepting the fact that the shock is adiabatic or nearly adiabatic requires that total energy is conserved, $T_{0x} = T_{0y}$. The relationship between the temperature and the stagnation temperature provides the relationship of the temperature for both sides of

the shock.

$$\frac{T_y}{T_x} = \frac{\frac{T_y}{T_{0y}}}{\frac{T_x}{T_{0x}}} = \frac{1 + \frac{k-1}{2} {M_x}^2}{1 + \frac{k-1}{2} {M_y}^2} \tag{6.7}$$

All the other relationships are essentially derived from this equation. The only issue left to derive is the relationship between M_x and M_y. Note that the Mach number is a function of temperature, and thus for known M_x all the other quantities can be determined, at least, numerically. The analytical solution is discussed in the next section.

6.1.2 Formal Model

Equations (6.1), (6.2), and (6.3) can be converted into a dimensionless form. The reason that dimensionless forms are heavily used in this book is because by doing so it simplifies and clarifies the solution. It can also be noted that in many cases the dimensionless equations set is more easily solved.

From the continuity equation (6.1) substituting for density, ρ, the equation of state yields

$$\frac{P_x}{R T_x} U_x = \frac{P_y}{R T_y} U_y \tag{6.8}$$

Squaring equation (6.8) results in

$$\frac{{P_x}^2}{R^2 {T_x}^2} {U_x}^2 = \frac{{P_y}^2}{R^2 {T_y}^2} {U_y}^2 \tag{6.9}$$

Multiplying the two sides by the ratio of the specific heat, k, provides a way to obtain the speed of sound definition/equation for perfect gas, $c^2 = k R T$ to be used for the Mach number definition, as follows:

$$\frac{{P_x}^2}{T_x \underbrace{k R T_x}_{{c_x}^2}} {U_x}^2 = \frac{{P_y}^2}{T_y \underbrace{k R T_y}_{{c_y}^2}} {U_y}^2 \tag{6.10}$$

Note that the speed of sound on the different sides of the shock is different. Utilizing the definition of Mach number results in

$$\frac{{P_x}^2}{T_x} {M_x}^2 = \frac{{P_y}^2}{T_y} {M_y}^2 \tag{6.11}$$

Rearranging equation (6.11) results in

$$\frac{T_y}{T_x} = \left(\frac{P_y}{P_x}\right)^2 \left(\frac{M_y}{M_x}\right)^2 \tag{6.12}$$

Energy equation (6.3) can be converted to a dimensionless form which can be expressed as

$$T_y \left(1 + \frac{k-1}{2} {M_y}^2\right) = T_x \left(1 + \frac{k-1}{2} {M_x}^2\right) \tag{6.13}$$

It can also be observed that equation (6.13) means that the stagnation temperature is the same, $T_{0y} = T_{0x}$. Under the perfect gas model, ρU^2 is identical to kPM^2 because

$$\rho U^2 = \overbrace{\frac{P}{RT}}^{\rho} \overbrace{\left(\frac{U^2}{\underbrace{k\,R\,T}_{c^2}}\right)}^{M^2} k\,R\,T = k\,P\,M2 \tag{6.14}$$

Using the identity (6.14) transforms the momentum equation (6.2) into

$$P_x + k\,P_x\,{M_x}^2 = P_y + k\,P_y\,{M_y}^2 \tag{6.15}$$

Rearranging equation (6.15) yields

$$\frac{P_y}{P_x} = \frac{1 + k\,{M_x}^2}{1 + k\,{M_y}^2} \tag{6.16}$$

The pressure ratio in equation (6.16) can be interpreted as the loss of the static pressure. The loss of the total pressure ratio can be expressed by utilizing the relationship between the pressure and total pressure (see equation (5.11)) as

$$\frac{P_{0y}}{P_{0x}} = \frac{P_y \left(1 + \frac{k-1}{2} {M_y}^2\right)^{\frac{k}{k-1}}}{P_x \left(1 + \frac{k-1}{2} {M_x}^2\right)^{\frac{k}{k-1}}} \tag{6.17}$$

The relationship between M_x and M_y is needed to be solved from the above set of equations. This relationship can be obtained from the combination of mass, momentum, and energy equations. From equation (6.13) (energy) and equation (6.12) (mass) the temperature ratio can be eliminated.

$$\left(\frac{P_y M_y}{P_x M_x}\right)^2 = \frac{1 + \frac{k-1}{2} {M_x}^2}{1 + \frac{k-1}{2} {M_y}^2} \tag{6.18}$$

Combining the results of (6.18) with equation (6.16) results in

$$\left(\frac{1 + k\,{M_x}^2}{1 + k\,{M_y}^2}\right)^2 = \left(\frac{M_x}{M_y}\right)^2 \frac{1 + \frac{k-1}{2} {M_x}^2}{1 + \frac{k-1}{2} {M_y}^2} \tag{6.19}$$

Equation (6.19) is a symmetrical equation in the sense that if M_y is substituted with M_x and M_x substituted with M_y the equation remains the same. Thus, one solution is

$$M_y = M_x \tag{6.20}$$

It can be observed that equation (6.19) is biquadratic. According to the Gauss Biquadratic Reciprocity Theorem this kind of equation has a real solution in a certain range[3] which will be discussed later. The solution can be obtained by rewriting equation (6.19) as a polynomial (fourth order). It is also possible to cross–multiply equation (6.19) and divide it by $\left({M_x}^2 - {M_y}^2\right)$ results in

$$1 + \frac{k-1}{2}\left({M_y}^2 + {M_y}^2\right) - k\,{M_y}^2\,{M_y}^2 = 0 \tag{6.21}$$

Equation (6.21) becomes

Shock Solution

$$M_y{}^2 = \frac{{M_x}^2 + \dfrac{2}{k-1}}{\dfrac{2\,k}{k-1}\,{M_x}^2 - 1} \tag{6.22}$$

The first solution (6.20) is the trivial solution in which the two sides are identical and no shock wave occurs. Clearly, in this case, the pressure and the temperature from both sides of the nonexistent shock are the same, i.e. $T_x = T_y,\ P_x = P_y$. The second solution is where the shock wave occurs.

The pressure ratio between the two sides can now be as a function of only a single Mach number, for example, M_x. Utilizing equation (6.16) and equation (6.22) provides the pressure ratio as only a function of the upstream Mach number as

$$\frac{P_y}{P_x} = \frac{2\,k}{k+1}{M_x}^2 - \frac{k-1}{k+1} \qquad \text{or}$$

Shock Pressure Ratio

$$\frac{P_y}{P_x} = 1 + \frac{2\,k}{k+1}\left({M_x}^2 - 1\right) \tag{6.23}$$

The density and upstream Mach number relationship can be obtained in the same fashion to became

Shock Density Ratio

$$\frac{\rho_y}{\rho_x} = \frac{U_x}{U_y} = \frac{(k+1){M_x}^2}{2 + (k-1){M_x}^2} \tag{6.24}$$

[3] Ireland, K. and Rosen, M. "Cubic and Biquadratic Reciprocity." Ch. 9 in A Classical Introduction to Modern Number Theory, 2nd ed. New York: Springer-Verlag, pp. 108-137, 1990.

The fact that the pressure ratio is a function of the upstream Mach number, M_x, provides additional way of obtaining an additional useful relationship. And the temperature ratio, as a function of pressure ratio, is transformed into

Shock Temperature Ratio

$$\frac{T_y}{T_x} = \left(\frac{P_y}{P_x}\right)\left(\frac{\frac{k+1}{k-1} + \frac{P_y}{P_x}}{1 + \frac{k+1}{k-1}\frac{P_y}{P_x}}\right) \tag{6.25}$$

In the same way, the relationship between the density ratio and pressure ratio is

Shock $P - \rho$

$$\frac{\rho_x}{\rho_y} = \frac{1 + \left(\frac{k+1}{k-1}\right)\left(\frac{P_y}{P_x}\right)}{\left(\frac{k+1}{k-1}\right) + \left(\frac{P_y}{P_x}\right)} \tag{6.26}$$

which is associated with the shock wave.

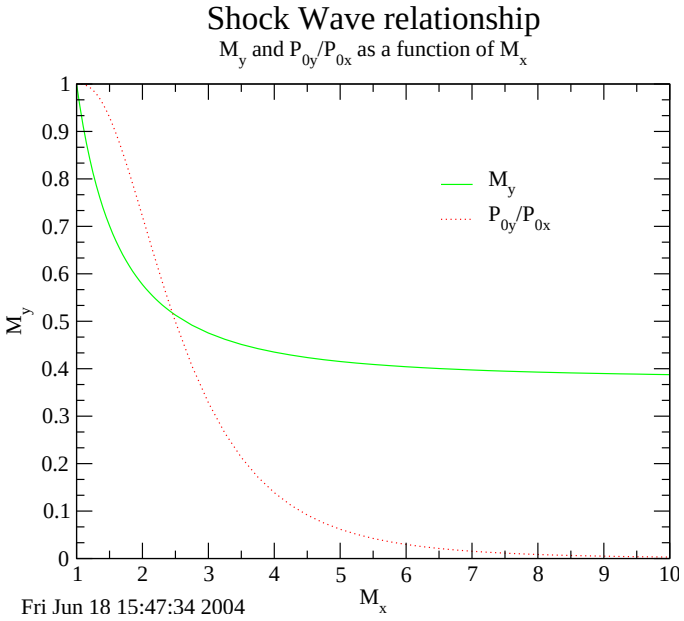


Fig. -6.3. The exit Mach number and the stagnation pressure ratio as a function of upstream Mach number.

The Maximum Conditions

The maximum speed of sound is when the highest temperature is achieved. The maximum temperature that can be achieved is the stagnation temperature

$$U_{max} = \sqrt{\frac{2\,k}{k-1}\,R\,T_0} \tag{6.27}$$

The stagnation speed of sound is

$$c_0 = \sqrt{k\,R\,T_0} \tag{6.28}$$

Based on this definition a new Mach number can be defined

$$M_0 = \frac{U}{c_0} \tag{6.29}$$

The Star Conditions

The speed of sound at the critical condition can also be a good reference velocity. The speed of sound at that velocity is

$$c^* = \sqrt{k\,R\,T^*} \tag{6.30}$$

In the same manner, an additional Mach number can be defined as

$$M^* = \frac{U}{c^*} \tag{6.31}$$

6.1.3 Prandtl's Condition

It can be easily observed that the temperature from both sides of the shock wave is discontinuous. Therefore, the speed of sound is different in these adjoining mediums. It is therefore convenient to define the star Mach number that will be independent of the specific Mach number (independent of the temperature).

$$M^* = \frac{U}{c^*} = \frac{c}{c^*}\frac{U}{c} = \frac{c}{c^*}\,M \tag{6.32}$$

The jump condition across the shock must satisfy the constant energy.

$$\frac{c^2}{k-1} + \frac{U^2}{2} = \frac{c^{*2}}{k-1} + \frac{c^{*2}}{2} = \frac{k+1}{2\,(k-1)}c^{*2} \tag{6.33}$$

Dividing the mass equation by the momentum equation and combining it with the perfect gas model yields

$$\frac{{c_1}^2}{k\,U_1} + U_1 = \frac{{c_2}^2}{k\,U_2} + U_2 \tag{6.34}$$

Combining equation (6.33) and (6.34) results in

$$\frac{1}{k\,U_1}\left[\frac{k+1}{2}\,c^{*2}-\frac{k-1}{2}U_1\right]+U_1=\frac{1}{k\,U_2}\left[\frac{k+1}{2}c^{*2}-\frac{k-1}{2}\,U_2\right]+U_2 \qquad (6.35)$$

After rearranging and dividing equation (6.35) the following can be obtained:

$$U_1\,U_2=c^{*2} \qquad (6.36)$$

or in a dimensionless form

$$M^*{}_1\,M^*{}_2=c^{*2} \qquad (6.37)$$

6.2 Operating Equations and Analysis

In Figure 6.3, the Mach number after the shock, M_y, and the ratio of the total pressure, P_{0y}/P_{0x}, are plotted as a function of the entrance Mach number. The working equations were presented earlier. Note that the M_y has a minimum value which depends on the specific heat ratio. It can be noticed that the density ratio (velocity ratio) also has a finite value regardless of the upstream Mach number.

The typical situations in which these equations can be used also include the moving shocks. The equations should be used with the Mach number (upstream or downstream) for a given pressure ratio or density ratio (velocity ratio). This kind of equations requires examining Table (6.2) for $k=1.4$ or utilizing Potto-GDC for for value of the specific heat ratio. Finding the Mach number for a pressure ratio of 8.30879 and $k=1.32$ and is only a few mouse clicks away from the following table.

To illustrate the use of the above equations, an example is provided.

Example 6.1:
Air flows with a Mach number of $M_x=3$, at a pressure of 0.5 [bar] and a temperature of 0° C goes through a normal shock. Calculate the temperature, pressure, total pressure, and velocity downstream of the shock. Assume that $k=1.4$.

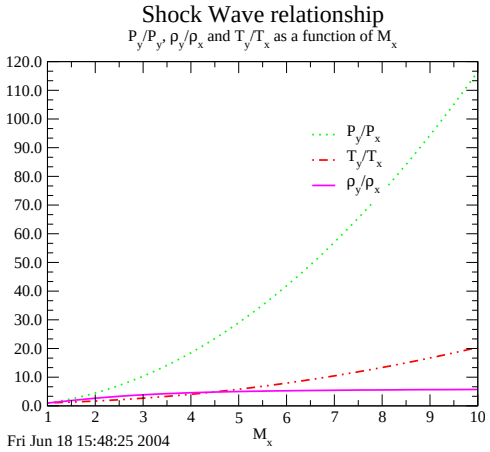


Fig. -6.4. The ratios of the static properties of the two sides of the shock.

SOLUTION

Analysis:
First, the known information are $M_x=3$, $P_x=1.5[bar]$ and $T_x=273K$. Using these data, the total pressure can be obtained (through an isentropic relationship in Table (5.2), i.e., P_{0x} is

known). Also with the temperature, T_x, the velocity can readily be calculated. The relationship that was calculated will be utilized to obtain the ratios for the downstream of the normal shock. $\frac{P_x}{P_{0x}} = 0.0272237 \Longrightarrow P_{0x} = 1.5/0.0272237 = 55.1[bar]$

$c_x = \sqrt{kRT_x} = \sqrt{1.4 \times 287 \times 273} = 331.2m/sec$

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.0000	0.47519	2.6790	3.8571	10.3333	0.32834

$U_x = M_x \times c_x = 3 \times 331.2 = 993.6[m/sec]$

Now the velocity downstream is determined by the inverse ratio of $\rho_y/\rho_x = U_x/U_y = 3.85714$.

$$U_y = 993.6/3.85714 = 257.6[m/sec]$$

$$P_{0y} = \left(\frac{P_{0y}}{P_{0x}}\right) \times P_{0x} = 0.32834 \times 55.1[bar] = 18.09[bar]$$

End Solution

6.2.1 The Limitations of the Shock Wave

When the upstream Mach number becomes very large, the downstream Mach number (see equation (6.22)) is limited by

$$M_y{}^2 = \frac{1 + \cancelto{\sim 0}{\frac{2}{(k-1)M_x{}^2}}}{\frac{2\,k}{k-1} - \cancelto{\sim 0}{\frac{1}{M_x{}^2}}} = \frac{k-1}{2\,k} \tag{6.38}$$

This result is shown in Figure 6.3. The limits of the pressure ratio can be obtained by looking at equation (6.16) and by utilizing the limit that was obtained in equation (6.38).

6.2.2 Small Perturbation Solution

The small perturbation solution refers to an analytical solution where only a small change (or several small changes) occurs. In this case, it refers to a case where only a "small shock" occurs, which is up to $M_x = 1.3$. This approach had a major significance and usefulness at a time when personal computers were not available. Now, during the writing of this version of the book, this technique is used mostly in obtaining analytical expressions for simplified models. This technique also has an academic value and therefore will be described in the next version (0.5.x series).

The strength of the shock wave is defined as

$$\hat{\mathcal{P}} = \frac{P_y - P_x}{P_x} = \frac{P_y}{P_x} - 1 \tag{6.39}$$

By using equation (6.23) transforms equation (6.39) into

$$\hat{\mathcal{P}} = \frac{2\,k}{k+1}\left({M_x}^2 - 1\right) \tag{6.40}$$

or by utilizing equation (6.24) the following is obtained:

$$\hat{\mathcal{P}} = \frac{\dfrac{2\,k}{k-1}\left(\dfrac{\rho_y}{\rho x} - 1\right)}{\dfrac{2}{k-1} - \left(\dfrac{\rho_y}{\rho_x} - 1\right)} \tag{6.41}$$

6.2.3 Shock Thickness

The issue of shock thickness (which will be presented in a later version) is presented here for completeness. This issue has a very limited practical application for most students; however, to convince the students that indeed the assumption of very thin shock is validated by analytical and experimental studies, the issue should be presented.

The shock thickness can be defined in several ways. The most common definition is by passing a tangent to the velocity at the center and finding out where the theoretical upstream and downstream conditions are meet.

6.2.4 Shock Drag or Wave Drag

It is communally believed that regardless to the cause of the shock, the shock creates a drag (due to increase of entropy). In this section, the first touch of this phenomenon will be presented. The fact that it is assumed that the flow is frictionless does not change whether or not shock drag occur. This explanation is broken into two sections: one for stationary shock wave, two for moving shock shock wave. A better explanation should appear in the oblique shock chapter.

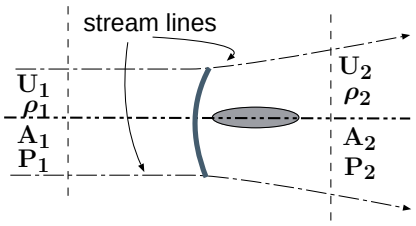


Fig. -6.5. The diagram that reexplains the shock drag effect.

Consider a normal shock as shown in figure (6.5). Gas flows in a supersonic velocity around a two–dimensional body and creates a shock. This shock is an oblique shock, however in this discussion, if the control volume is chosen close enough to the body is can be considered as almost a normal shock (in the oblique shock chapter a section on this issue will be presented that explains the fact that shock is oblique, to be irrelevant).

The control volume that is used here is along two stream lines. The other two boundaries are arbitrary but close enough to the body. Along the stream lines there is no mass exchange and therefore there is no momentum exchange. Moreover, it is

assumed that the gas is frictionless, therefore no friction occurs along any stream line. The only change is two arbitrary surfaces since the pressure, velocity, and density are changing. The velocity is reduced $U_x > U_y$. However, the density is increasing, and in addition, the pressure is increasing. So what is the momentum net change in this situation? To answer this question, the momentum equation must be written and it will be similar to equation (5.103). However, since $\frac{F_y}{F^*} = \frac{F_x}{F^*}$ there is no net force acting on the body. For example, consider upstream of $M_x = 3.$ and for which

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{Po_y}}{\mathbf{Po_x}}$
3.0000	0.47519	2.6790	3.8571	10.3333	0.32834

and the corespondent Isentropic information for the Mach numbers is

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
3.0000	0.35714	0.07623	4.2346	0.02722	0.11528	0.65326
0.47519	0.95679	0.89545	1.3904	0.85676	1.1912	0.65326

Now, after it was established, it is not a surprising result. After all, the shock analysis started with the assumption that no momentum is change. As conclusion there is no shock drag at stationary shock. This is not true for moving shock as it will be discussed in section (6.3.1).

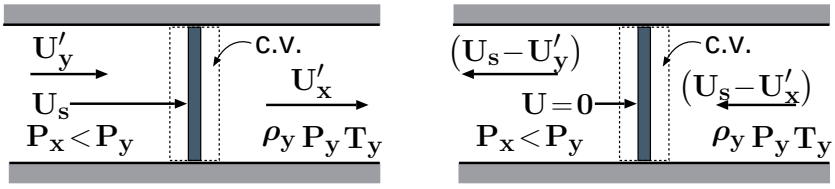
6.3 The Moving Shocks

In some situations, the shock wave is not stationary. This kind of situation arises in many industrial applications. For example, when a valve is suddenly [4] closed and a shock propagates upstream. On the other extreme, when a valve is suddenly opened or a membrane is ruptured, a shock occurs and propagates downstream (the opposite direction of the previous case). In addition to (partially) closing or (partially) opening of value, the rigid body (not so rigid body) movement creates shocks. In some industrial applications, a liquid (metal) is pushed in two rapid stages to a cavity through a pipe system. This liquid (metal) is pushing gas (mostly) air, which creates two shock stages. The moving shock is observed by daily as hearing sound wave are moving shocks.

As a general rule, the moving shock can move downstream or upstream. The source of the shock creation, either due to the static device operation like valve operating/closing or due to moving object, is relevant to analysis but it effects the boundary conditions. This creation difference while creates the same moving shock it creates different questions and hence in some situations complicate the calculations. The most general case which this section will be dealing with is the partially open or close wave.

[4] It will be explained using dimensional analysis what is suddenly open.

A brief discussion on the such case (partially close/open but due the moving object) will be presented. There are more general cases where the moving shocks are created which include a change in the physical properties, but this book will not deal with them at this stage. The reluctance to deal with the most general case is due to fact it is highly specialized and complicated even beyond early graduate students level. In these changes (of opening a valve and closing a valve on the other side) create situations in which different shocks are moving in the tube. The general case is where two shocks collide into one shock and moves upstream or downstream is the general case. A specific example is common in die–casting: after the first shock moves a second shock is created in which its velocity is dictated by the upstream and downstream velocities.



(a) Stationary coordinates. (b) Moving coordinates.

Fig. -6.6. Comparison between stationary and moving coordinates for the moving shock.

In cases where the shock velocity can be approximated as a constant (in the majority of cases) or as near constant, the previous analysis, equations, and the tools developed in this chapter can be employed. The problem can be reduced to the previously studied shock, i.e., to the stationary case when the coordinates are attached to the shock front. In such a case, the steady state is obtained in the moving control value.

For this analysis, the coordinates move with the shock. Here, the prime ' denotes the values of the static coordinates. Note that this notation is contrary to the conventional notation found in the literature. The reason for the deviation is that this choice reduces the programing work (especially for object–oriented programing like C++). An observer moving with the shock will notice that the pressure in the shock sides is

$$P_x{}' = P_x \qquad P_y{}' = P_y \tag{6.42}$$

The temperatures measured by the observer are

$$T_x{}' = T_x \qquad T_y{}' = T_y \tag{6.43}$$

Assuming that the shock is moving to the right, (refer to Figure 6.6) the velocity measured by the observer is

$$U_x = U_s - U_x{}' \tag{6.44}$$

Where U_s is the shock velocity which is moving to the right. The "downstream" velocity is

$$U_y{}' = U_s - U_y \tag{6.45}$$

The speed of sound on both sides of the shock depends only on the temperature and it is assumed to be constant. The upstream prime Mach number can be defined as

$$M_x{}' = \frac{U_s - U_x}{c_x} = \frac{U_s}{c_x} - M_x = M_{sx} - M_x \tag{6.46}$$

It can be noted that the additional definition was introduced for the shock upstream Mach number, $M_{sx} = \frac{U_s}{c_x}$. The downstream prime Mach number can be expressed as

$$M_y{}' = \frac{U_s - U_y}{c_y} = \frac{U_s}{c_y} - M_y = M_{sy} - M_y \tag{6.47}$$

Similar to the previous case, an additional definition was introduced for the shock downstream Mach number, M_{sy}. The relationship between the two new shock Mach numbers is

$$\begin{aligned} \frac{U_s}{c_x} &= \frac{c_y}{c_x}\frac{U_s}{c_y} \\ M_{sx} &= \sqrt{\frac{T_y}{T_x}}\, M_{sy} \end{aligned} \tag{6.48}$$

The "upstream" stagnation temperature of the fluid is

Shock Stagnation Temperature

$$T_{0x} = T_x\left(1 + \frac{k-1}{2} {M_x}^2\right) \tag{6.49}$$

and the "upstream" prime stagnation pressure is

$$P_{0x} = P_x\left(1 + \frac{k-1}{2} {M_x}^2\right)^{\frac{k}{k-1}} \tag{6.50}$$

The same can be said for the "downstream" side of the shock. The difference between the stagnation temperature is in the moving coordinates

$$T_{0y} - T_{0x} = 0 \tag{6.51}$$

It should be noted that the stagnation temperature (in the stationary coordinates) rises as opposed to the stationary normal shock. The rise in the total temperature is due to the fact that a new material has entered the c.v. at a very high velocity, and is

"converted" or added into the total temperature,

$$\begin{aligned} T_{0y} - T_{0x} =& T_y \left(1 + \frac{k-1}{2}\left(M_{sy} - M_y{}'\right)^2\right) - T_x \left(1 + \frac{k-1}{2}\left(M_{sx} - M_x{}'\right)^2\right) \\ 0 =& \overbrace{T_y \left(1 + \frac{k-1}{2} M_y{}'^2\right)}^{T_{0y}{}'} + T_y M_{sy} \frac{k-1}{2}\left(M_{sy} - 2M_y\right) \\ & - \overbrace{T_x \left(1 + \frac{k-1}{2} M_x{}'^2\right)}^{T_{0x}{}'} - T_x M_{sx} \frac{k-1}{2}\left(M_{sx} - 2M_x\right) \end{aligned} \tag{6.52}$$

and according to equation (6.51) leads to

$$T_{0y}{}' - T_{0x}{}' = U_s \left(\frac{T_x}{c_x}\frac{k-1}{2}\left(M_{sx} - 2M_x\right) - \frac{T_y}{c_y}\frac{k-1}{2}\left(M_{sy} - 2M_y\right)\right) \tag{6.53}$$

Again, this difference in the moving shock is expected because moving material velocity (kinetic energy) is converted into internal energy. This difference can also be viewed as a result of the unsteady state of the shock.

6.3.1 Shock or Wave Drag Result from a Moving Shock

In section (6.2.4) it was shown that there is no shock drag in stationary shock. However, the shock or wave drag is very significant so much so that at one point it was considered the sound barrier. Consider the figure (6.7) where the stream lines are moving with the object speed. The other boundaries are stationary but the velocity at right boundary is not zero. The same arguments, as discussed before in the stationary case, are applied. What is different in the present case (as oppose to the stationary shock), one side has increase the momentum of the control volume. This increase momentum in the control volume causes the shock drag. In way, it can be view as continuous acceleration of the gas around the body from zero. Note this drag is only applicable to a moving shock (unsteady shock).

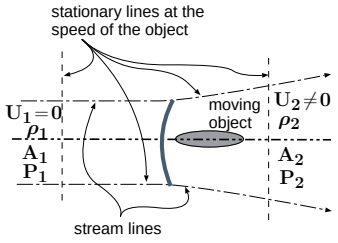


Fig. -6.7. The diagram that reexplains the shock drag effect of a moving shock.

The moving shock is either results from a body that moves in gas or from a sudden imposed boundary like close or open valve[5]. In the first case, the forces or energies flow from body to gas and therefor there is a need for large force to accelerate the gas over extremely short distance (shock thickness). In the second case, the gas contains the

[5]According to my son, the difference between these two cases is the direction of the information. Both case there essentially bodies, however, in one the information flows from inside the field to the boundary while the other case it is the opposite.

energy (as high pressure, for example in the open valve case) and the energy potential is lost in the shock process (like shock drag).

For some strange reasons, this topic has several misconceptions that even appear in many popular and good textbooks[6]. Consider the following example taken from such a book.

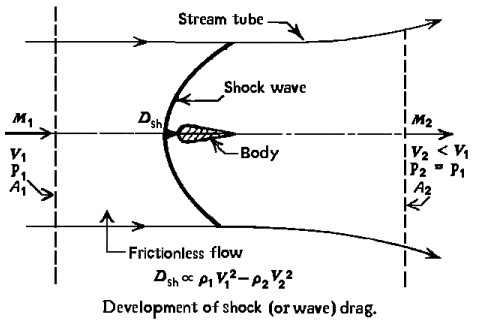


Fig. -6.8. The diagram for the common explanation for shock or wave drag effect a shock. Please notice the strange notations (e.g. V and not U) and they result from a verbatim copy.

Example 6.2:
A book (see Figure 6.8) explains the shock drag is based on the following rational: The body is moving in a stationary frictionless fluid under one–dimensional flow. The left plane is moving with body at the same speed. The second plane is located "downstream from the body where the gas has expanded isotropically (after the shock wave) to the upstream static pressure". the bottom and upper stream line close the control volume. Since the pressure is the same on the both planes there is no unbalanced pressure forces. However, there is a change in the momentum in the flow direction because $U_1 > U_2$. The force is acting on the body. There several mistakes in this explanation including the drawing. Explain what is wrong in this description (do not describe the error results from oblique shock).

SOLUTION

Neglecting the mistake around the contact of the stream lines with the oblique shock(see for retouch in the oblique chapter), the control volume suggested is stretched with time. However, the common explanation fall to notice that when the isentropic explanation occurs the width of the area change. Thus, the simple explanation in a change only in momentum (velocity) is not appropriate. Moreover, in an expanding control volume this simple explanation is not appropriate. Notice that the relative velocity at the front

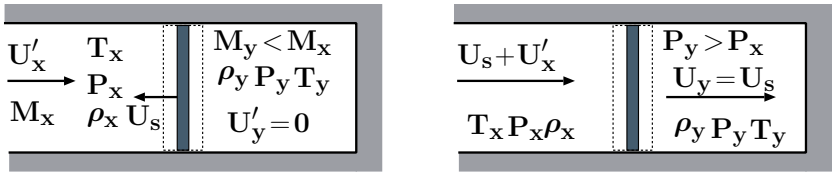
[6]Similar situation exist in the surface tension area.

of the control volume U_1 is actually zero. Hence, the claim of $U_1 > U_2$ is actually the opposite, $U_1 < U_2$.

End Solution

6.3.2 Shock Result from a Sudden and Complete Stop

The general discussion can be simplified in the extreme case when the shock is moving from a still medium. This situation arises in many cases in the industry, for example, in a sudden and complete closing of a valve. The sudden closing of the valve must result in a zero velocity of the gas. This shock is viewed and referred by some as a reflective shock. The information propagates upstream in which the gas velocity is converted into temperature. In many such cases the quasi steady state is established quite rapidly. In such a case, the shock velocity "downstream" is U_s. Equations (6.42) to (6.53) can be transformed into simpler equations when M_x is zero and U_s is a positive value. Figure 6.9 shows the schematic in two different coordinates for the analysis below.



(a) Stationary coordinates. (b) Moving coordinates.

Fig. -6.9. Comparison between shock results from the close valve in two stationary and moving coordinates.

The "upstream" Mach number reads

$$M_x = \frac{U_s + U_x}{c_x} = M_{sx} + M_x \tag{6.54}$$

The "downstream" Mach number reads

$$M_y = \frac{|U_s|}{c_y} = M_{sy} \tag{6.55}$$

Again, the shock is moving to the left. In the moving coordinates, the observer (with the shock) sees the flow moving from the left to the right. The flow is moving to the right. The upstream is on the left of the shock. The stagnation temperature increases by

$$T_{0y} - T_{0x} = U_s \left(\frac{T_x}{c_x} \frac{k-1}{2} \left(M_{sx} + 2M_x \right) - \frac{T_y}{c_y} \frac{k-1}{2} \left(M_{sy} \right) \right) \tag{6.56}$$

The prominent question in this situation is what will be the shock wave velocity for a given fluid velocity, $U_x{}'$, and for a given specific heat ratio. The "upstream" or

the "downstream" Mach number is not known even if the pressure and the temperature downstream are given. The difficulty lies in the jump from the stationary coordinates to the moving coordinates. It turns out that it is very useful to use the dimensionless parameter M_{sx}, and or M_{sy} instead of the velocity because it combines the temperature and the velocity into one parameter.

The relationship between the Mach number on the two sides of the shock are tied through equations (6.54) and (6.55) by

$$(M_y)^2 = \frac{\left(M_x{}' + M_{sx}\right)^2 + \frac{2}{k-1}}{\frac{2\,k}{k-1}\left(M_x{}' + M_{sx}\right)^2 - 1} \tag{6.57}$$

And substituting equation (6.57) into (6.48) results in

$$M_x = \overbrace{\sqrt{\frac{T_x}{T_y}}}^{f(M_{sx})} \sqrt{\frac{\left(M_x{}' + M_{sx}\right)^2 + \frac{2}{k-1}}{\frac{2k}{k-1}\left(M_x{}' + M_{sx}\right)^2 - 1}} \tag{6.58}$$

The temperature ratio in equation (6.58) and the rest of the right–hand side show clearly that M_{sx} has four possible solutions (fourth–order polynomial M_{sx} has four solutions). Only one real solution is possible. The solution to equation (6.58) can be obtained by several numerical methods. Note, an analytical solution can be obtained for equation (6.58) but it seems utilizing numerical methods is much more simple. The typical method is the "smart" guessing of M_{sx}. For very small values of the upstream Mach number, $M_x{}' \sim \epsilon$ equation (6.58) provides that $M_{sx} \sim 1+\frac{1}{2}\epsilon$ and $M_{sy} = 1-\frac{1}{2}\epsilon$ (the coefficient is only approximated as 0.5) as shown in Figure 6.10. From the same figure it can also be observed that a high velocity can result in a much larger velocity for the reflective shock. For example, a Mach number close to one (1), which can easily be obtained in a Fanno flow, the result is about double the sonic velocity of the reflective shock. Sometimes this phenomenon can have a tremendous significance in industrial applications.

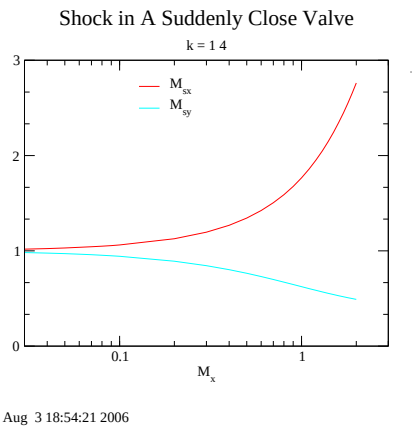


Fig. -6.10. The moving shock Mach numbers as a result of a sudden and complete stop.

Note that to achieve a supersonic velocity (in stationary coordinates) a diverging–converging nozzle is required. Here no such device is needed! Luckily and hopefully,

engineers who are dealing with a supersonic flow when installing the nozzle and pipe systems for gaseous mediums understand the importance of the reflective shock wave.

Two numerical methods and or algorithm employed to solve this problem for given, $M_x{}'$, is provided herein:

Method 1:

(a) Guess $M_x > 1$,

(b) Using shock table or use Potto–GDC to calculate temperature ratio and M_y,

(c) Calculate the $M_x = M_x{}' - \sqrt{\frac{T_x}{T_y}} M_y$

(d) Compare to the calculated $M_x{}'$ to the given $M_x{}'$. and adjust the new guess $M_x > 1$ accordingly.

Mehod 2:

The second method is "successive substitutions," which has better convergence to the solution initially in most ranges but less effective for higher accuracies.

(a) Guess $M_x = 1 + M_x{}'$,

(b) using the shock table or use Potto–GDC to calculate the temperature ratio and M_y,

(c) calculate the $M_x = M_x{}' - \sqrt{\frac{T_x}{T_y}} M_y$

(d) Compare the new M_x approach the old M_x, if not satisfactory use the new $M_x{}'$ to calculate $M_x = 1 + M_x{}'$ then return to part (b).

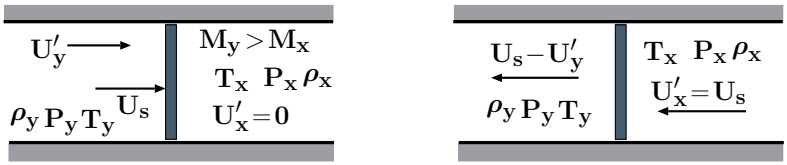
6.3.3 Moving Shock into Stationary Medium (Suddenly Open Valve)

General Velocities Issues

When a valve or membrane is suddenly opened, a shock is created and propagates downstream. With the exception of close proximity to the valve, the shock moves in a constant velocity (6.11(a)). Using a coordinates system which moves with the shock results in a stationary shock and the flow is moving to the left see Figure 6.11(b). The "upstream" will be on the right (see Figure 6.11(b)).

Similar definitions of the right side and the left side of the shock Mach numbers can be utilized. It has to be noted that the "upstream" and "downstream" are the reverse from the previous case. The "upstream" Mach number is

$$M_x = \frac{U_s}{c_x} = M_{sx} \tag{6.59}$$



(a) Stationary coordinates. (b) Moving coordinates.

Fig. -6.11. A shock moves into a still medium as a result of a sudden and complete opening of a valve.

The "downstream" Mach number is

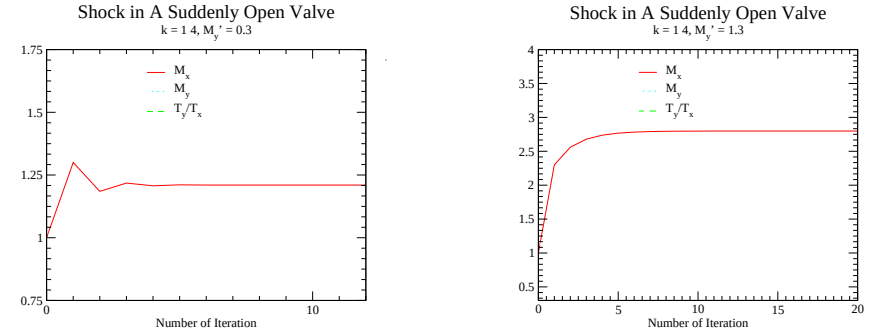
$$M_y = \frac{U_s - U_y{}'}{c_y} = M_{sy} - M_y{}' \tag{6.60}$$

Note that in this case the stagnation temperature in stationary coordinates changes (as in the previous case) whereas the thermal energy (due to pressure difference) is converted into velocity. The stagnation temperature (of moving coordinates) is

$$T_{0y} - T_{0x} = T_y \left(1 + \frac{k-1}{2} \left(M_{sy} - M_y\right)^2\right) - T_x \left(1 + \frac{k-1}{2} \left(M_x\right)^2\right) = 0 \tag{6.61}$$

A similar rearrangement to the previous case results in

$$T_{0y}{}' - T_{0x}{}' = T_y \left(1 + \frac{k-1}{2} \left(-2M_{sy}M_y + M_y{}^2\right)^2\right) \tag{6.62}$$



(a) $M_y{}' = 0.3$ (b) $M_y{}' = 1.3$

Fig. -6.12. The number of iterations to achieve convergence.

The same question that was prominent in the previous case appears now, what will be the shock velocity for a given upstream Mach number? Again, the relationship

between the two sides is

$$M_{sy} = M_y{}' + \sqrt{\frac{(M_{sx})^2 + \frac{2}{k-1}}{\frac{2k}{k-1}(M_{sx})^2 - 1}} \tag{6.63}$$

Since M_{sx} can be represented by M_{sy} theoretically equation (6.63) can be solved. It is common practice to solve this equation by numerical methods. One such methods is "successive substitutions." This method is applied by the following algorithm:

(a) Assume that $M_x = 1.0$.

(b) Calculate the Mach number M_y by utilizing the tables or Potto–GDC.

(c) Utilizing

$$M_x = \sqrt{\frac{T_y}{T_x}}\left(M_y + M_y{}'\right)$$

calculate the new "improved" M_x.

(d) Check the new and improved M_x against the old one. If it is satisfactory, stop or return to stage (b).

To illustrate the convergence of the procedure, consider the case of $M_y{}' = 0.3$ and $M_y{}' = 0.3$. The results show that the convergence occurs very rapidly (see Figure 6.12). The larger the value of $M_y{}'$, the larger number of the iterations required to achieve the same accuracy. Yet, for most practical purposes, sufficient results can be achieved after 3-4 iterations for given pressure ratio.

Piston Velocity When a piston is moving, it creates a shock that moves at a speed greater than that of the piston itself. The unknown data are the piston velocity, the temperature, and, other conditions ahead of the shock. Therefore, no Mach number is given but pieces of information on both sides of the shock. In this case, the calculations for U_s can be obtained from equation (6.24) that relate the shock velocities and Shock Mach number as

$$\frac{U_x}{U_y} = \frac{M_{sx}}{M_{sx} - \frac{U_y{}'}{c_x}} = \frac{(k+1){M_{sx}}^2}{2 + (k-1){M_{sx}}^2} \tag{6.64}$$

Equation (6.64) is a quadratic equation for M_{sx}. There are three solutions of which the first one is $M_{sx} = 0$ and this is immediately disregarded. The other two solutions are

$$M_{sx} = \frac{(k+1)U_y{}' \pm \sqrt{\left[U_y{}'(1+k)\right]^2 + 16{c_x}^2}}{4\ c_x} \tag{6.65}$$

The negative sign provides a negative value which is disregarded, and the only solution left is

$$M_{sx} = \frac{(k+1)U_y{}' + \sqrt{\left[U_y{}'(1+k)\right]^2 + 16{c_x}^2}}{4\ c_x} \tag{6.66}$$

or in a dimensionless form

Piston Moving Shock

$$M_{sx} = \frac{(k+1)M_{yx}{}' + \sqrt{\left[M_{yx}{}'(1+k)\right]^2 + 16}}{4} \tag{6.67}$$

Where the "strange," Mach number is $M_{sx}{}' = U_y{}'/c_x$. The limit of the equation when $c_x \to \infty$ leads to

$$M_{sx} = \frac{(k+1)M_{yx}{}'}{4} \tag{6.68}$$

As one additional "strange" it can be seen that the shock is close to the piston when the gas ahead of the piston is very hot. This phenomenon occurs in many industrial applications, such as the internal combustion engines and die casting. Some use equation (6.68) to explain the (to be discussed) Shock–Choke phenomenon.

In one of the best book in fluid mechanics provides a problem that is the similar to the piston pushing but with a twist. In this section analysis will carried for the error in neglecting the moving shock. This problem is discussed here because at first glance looks a simple problem, however, the physics of the problem is a bit complicated and deserve a discussion[7].

A piston with a known dimensions (shown in Figure 6.13 is pushed by a constant force. The gas (air) with an initial temperature is pushed through a converging nozzle (shown in the original schematic). The point where the moving shock reaches to the exit there are two situations:choked and unchoked flow. If the flow is choked, then the Mach number at the exit is one. If the flow is unchoked, then the exit Mach number is unknown but the pressure ratio is know. Assuming the flow is choked (see later for the calculation) the exit Mach number is 1 and therefor, $U_e = \sqrt{k\,RT} = \sqrt{1.4 \times 287 \times 0.833 \times 293.15} \sim 313[m/sec]$. The velocity at the cylinder is assumed to be isentropic and hence area ratio is $A/A^* = 1600$ the condition at the cylinder can be obtained from Potto-GDC as

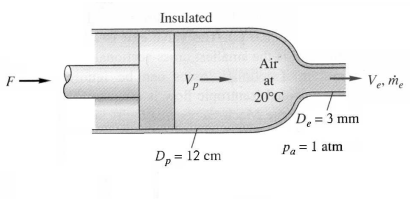


Fig. -6.13. Schematic of showing the piston pushing air.

[7] A student from France forward this problem to this author after argument with his instructor. The instructor used the book's manual solution and refused to accept the student improved solution which he learned from this book/author. Therefore, this problem will be referred as the French problem.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
$3.614E{-}4$	1.0	1.0	$1.6E{+}3$	1.0	$1.6E{+}3$	$6.7E{+}2$

The piston velocity is then $U_{piston} = 0.000361425 \times \sqrt{1.4 \times 287 \times 297.15} \sim 0.124[m/sec]$.

Before the semi state state is achieved, the piston is accelerated to the constant velocity (or at least most constant velocity). A this stage, a shock wave is moving away from piston toward the nozzle. If this shock reaches to exit before the semi state is achieved, the only way to solve this problem is by a numerical method (either characteristic methods or other numerical method) and it is out of the scope of this chapter. The transition of the moving shock through the converging nozzle is neglected in this discussion. However, if a quasi steady state is obtained, this discussion deals with that case. Before the shock is reaching to exit no flow occur at the exit (as opposite to the solution which neglects the moving shock).

The first case (choked, which is the more common, for example, syringe when pushing air has similar situations), is determined from the fact that pressure at the cylinder can be calculated. If the pressure ratio is equal or higher than the critical ratio then the flow is choked. For the unchoked case, the exit Mach number is unknown. However, the pressure ratio between the cylinder and the outside world is known. The temperature in the cylinder has to be calculated using moving shock relationship.

In the present case, the critical force should be calculated first. The specific heat ratio is $k = 1.4$ and therefore critical pressure ratio is 0.528282. The critical force is

$$F_{critical} = P_{critical} A_{piston} = P_a \frac{P_{critical}}{P_a} A_{piston} \tag{6.69}$$

In this case

$$F_{critical} = 101325(1/0.528282 - 1) \times \frac{\pi \times 0.12^2}{4} \sim 1022.74[N]$$

Since the force is 1100 [N], it is above the critical force the flow is chocked. The pressure ratio between the cylinder and the choking point is the critical pressure ratio. It should be noted that further increase of the force will not change the pressure ratio but the pressure at the choking point.

$$\frac{P_{cylinder}}{P_a} = \frac{101325 + \frac{1100}{\frac{\pi \times 0.12^2}{4}}}{101325} = 1.96$$

The moving shock conditions are determined from the velocity of the piston. As first approximation the piston Mach number is obtained from the area ratio in isentropic flow ($3.614E^{-4}$). Using this Mach number is $M_y{}'$ Potto-GDC provides

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.0002	0.99978	0.0	0.000361	1.0	1.001	1.0

The improved the piston pressure ratio ("piston" pressure to the nozzle pressure) is changed by only 0.1%. Improved accuracy can be obtained in the second iteration by taking this shock pressure ratio into consideration. However, here, for most engineering propose this improvement is insignificant. This information provides the ability to calculate the moving shock velocity.

$$V_{shock} = c\,M_s = c\,M_x = 1.0002\sqrt{1.4 \times 287 \times 293.15} \sim 343.3[m/sec]$$

The time for the moving shock to reach depends on the length of the cylinder as

$$t = \frac{L_{cylinder}}{V_{shock}} \tag{6.70}$$

For example, in case the length is three times the diameter will result then the time is

$$t = \frac{3 \times 0.12}{343.3} \sim 0.001[sec]$$

In most case this time is insignificant, however, there are process and conditions that this shock affects the calculations. In Figure 6.14 shows the pressure at the nozzle and the piston velocity. It can be observed that piston velocity increase to constant velocity very fast. Initially the transition continue until a quasi steady state is obtained. This quasi steady state continues until the shock reaches to the nuzzle and the pressure at the nozzle jump in a small amount (see Figure 6.14).

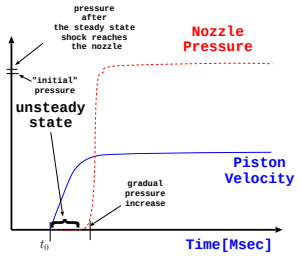


Fig. -6.14. Time the pressure at the nozzle for the French problem.

Shock–Choke Phenomenon

Assuming that the gas velocity is supersonic (in stationary coordinates) before the shock moves, what is the maximum velocity that can be reached before this model fails? In other words, is there a point where the moving shock is fast enough to reduce the "upstream" relative Mach number below the speed of sound? This is the point where regardless of the pressure difference is, the shock Mach number cannot be increased.

This shock–choking phenomenon is somewhat similar to the choking phenomenon that was discussed earlier in a nozzle flow and in other pipe flow models (later chapters). The difference is that the actual velocity has no limit. It must be noted that in the previous case of suddenly and completely closing of valve results in no limit (at least from the model point of view). To explain this phenomenon, look at the normal shock. Consider when the "upstream" Mach approaches infinity, $M_x = M_{sx} \to \infty$, and the downstream Mach number, according to equation (6.38), is approaching to $(k-1)/2k$. One can view this as the source of the shock–choking phenomenon. These

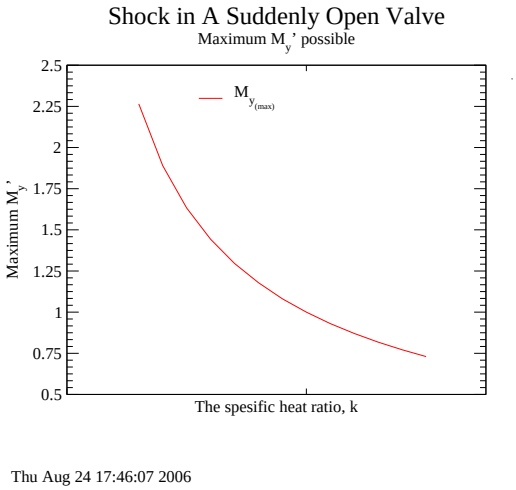


Fig. -6.15. The maximum of "downstream" Mach number as a function of the specific heat, k.

limits determine the maximum velocity after the shock since $U_{max} = c_y M_y$. From the upstream side, the Mach number is

$$M_x = M_{sx} = \sqrt{\frac{T_y}{T_x}}^{\infty} \left(\frac{k-1}{2k} \right) \tag{6.71}$$

Thus, the Mach number is approaching infinity because of the temperature ratio but the velocity is finite.

To understand this limit, consider that the maximum Mach number is obtained when the pressure ratio is approaching infinity $\frac{P_y}{P_x} \to \infty$. By applying equation (6.23) to this situation the following is obtained:

$$M_{sx} = \sqrt{\frac{k+1}{2k} \left(\frac{P_x}{P_y} - 1 \right) + 1} \tag{6.72}$$

and the mass conservation leads to

$$\begin{aligned} U_y \rho_y &= U_s \rho_x \\ \left(U_s - U_y{}' \right) \rho_y &= U_s \rho_x \\ M_y{}' &= \sqrt{\frac{T_y}{T_x}} \left(1 - \frac{\rho_x}{\rho_y} \right) M_{sx} \end{aligned} \tag{6.73}$$

Substituting equations (6.26) and (6.25) into equation (6.73) results in

$$M_y{}' = \frac{1}{k}\left(1-\frac{P_y}{P_x}\right)\sqrt{\frac{\frac{2k}{k+1}}{\frac{P_y}{P_x}+\frac{k-1}{k+1}}}\times\sqrt{\frac{1+\left(\frac{k+1}{k-1}\right)\left(\frac{P_y}{P_x}\right)}{\left(\frac{k+1}{k-1}\right)+\left(\frac{P_y}{P_x}\right)}} \tag{6.74}$$

When the pressure ratio is approaching infinity (extremely strong pressure ratio), the results is

$$M_y{}' = \sqrt{\frac{2}{k(k-1)}} \tag{6.75}$$

What happens when a gas with a Mach number larger than the maximum Mach number possible is flowing in the tube? Obviously, the semi steady state described by the moving shock cannot be sustained. A similar phenomenon to the choking in the nozzle and later in an internal pipe flow is obtained. The Mach number is reduced to the maximum value very rapidly. The reduction occurs by an increase of temperature after the shock or a stationary shock occurs as it will be shown in chapters on internal flow.

$\mathbf{k}$	$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$
1.30	1073.25	0.33968	2.2645	169842.29
1.40	985.85	0.37797	1.8898	188982.96
1.50	922.23	0.40825	1.6330	204124.86
1.60	873.09	0.43301	1.4434	216507.05
1.70	833.61	0.45374	1.2964	226871.99
1.80	801.02	0.47141	1.1785	235702.93
1.90	773.54	0.48667	1.0815	243332.79
2.00	750.00	0.50000	1.00000	250000.64
2.10	729.56	0.51177	0.93048	255883.78
2.20	711.62	0.52223	0.87039	261117.09
2.30	695.74	0.53161	0.81786	265805.36
2.40	681.56	0.54006	0.77151	270031.44
2.50	668.81	0.54772	0.73029	273861.85

Table -6.1. Table of maximum values of the shock-choking phenomenon.

The mass flow rate when the pressure ratio is approaching infinity, ∞, is

$$\frac{\dot{m}}{A} = {U_y}' \rho_y = {M_y}' c_y \rho_y = {M_y}' \overbrace{\sqrt{k\,R\,T_y}}^{c_y} \overbrace{\frac{P_y}{RT_y}}^{\rho_y}$$
$$= \frac{{M_y}' \sqrt{k} P_y}{\sqrt{RT_y}} \tag{6.76}$$

Equation (6.76) and equation (6.25) can be transferred for large pressure ratios into

$$\frac{\dot{m}}{A} \sim \sqrt{T_y} \frac{P_x}{T_x} \frac{k-1}{k+1} \tag{6.77}$$

Since the right hand side of equation (6.77) is constant, with the exception of $\sqrt{T_y}$ the mass flow rate is approaching infinity when the pressure ratio is approaching infinity. Thus, the shock–choke phenomenon means that the Mach number is only limited in stationary coordinates but the actual flow rate isn't.

Moving Shock in Two and Three Dimensions

A moving shock into a still gas can occur in a cylindrical or a spherical coordinates[8]. For example, explosion can be estimated as a shock moving in a three dimensional direction in uniform way. A long line of explosive can create a cylindrical moving shock. These shocks are similar to one dimensional shock in which a moving gas is entering a still gas. In one dimensional shock the velocity of the shock is constant. In two and three dimensions the pressure and shock velocity and velocity behind the shock are function of time. These difference decrease the accuracy of the calculation because the unsteady part is not accounted for. However, the gain is the simplicity of the calculations. The relationships that have been developed so far for the normal shock are can be used for this case because the shock is perpendicular to the flow. However, it has to be remembered that for very large pressure difference the unsteadiness has to be accounted. The deviation increases as the pressure difference decrease and the geometry became larger. Thus, these result provides the limit for the unsteady state. This principle can be demonstrated by looking in the following simple example.

Example 6.3:
After sometime after an explosion a spherical "bubble" is created with pressure of $20[Bar]$. *Assume that the atmospheric pressure is* $1[Bar]$ *and temperature of* $27^{\circ}\mathrm{C}$ *Estimate the higher limit of the velocity of the shock, the velocity of the gas inside the "bubble" and the temperature inside the bubble. Assume that* $k = 1.4$ *and* $R = 287[j/kg/K$ *and no chemical reactions occur.*

[8]Dr. Attiyerah asked me to provide example for this issue. Explosion is not my area of research but it turned to be similar to the author's work on evacuation and filling of semi rigid chambers. It also similar to shock tube and will be expanded later.

SOLUTION

The Mach number can be estimated from the pressure ratio

$$\frac{P_{inside}}{P_{outside}} = 20$$

. One can obtain using Potto–gdc the following

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{Po_y}}{\mathbf{Po_x}}$
4.1576	0.43095	4.2975	4.6538	20.0000	0.12155

or by using the shock dynamics section the following

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{Po_y}}{\mathbf{Po_x}}$
4.1576	0.43095	0	1.575	4.298	20	0.12155

The shock velocity estimate is then

$$U_s = \overbrace{M_s}^{M_x} c_y = 4.1576 \times \sqrt{1.4 \times 287 \times 300} \sim 1443.47[m/sec]$$

The temperature inside the "bubble" is then

$$T_y = \frac{T_y}{T_x} T_x = 4.298 \times 300 \sim 1289.4K$$

The velocity of the gas inside the "bubble" is then

$$U_y{}' = M_y{}' c_y = 1.575 \times \sqrt{1.4 \times 287 \times 1289.4} \sim 1133.65[m/sec]$$

These velocities estimates are only the upper limits. The actual velocity will be lower due to the unsteadiness of the situation.

End Solution

This problem is unsteady state but can be considered as a semi steady state. This kind of analysis creates a larger error but gives the trends and limits. The common problem is that for a given pressure ratio and initial radius (volume) the shock velocity and inside gas velocity inside are needed. As first approximation it can be assumed material inside the "bubble' is uniform and undergoes isentropic process. This is similar to shock tube.

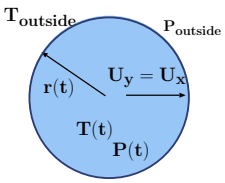


Fig. -6.16. Time the pressure at the nozzle for the French problem.

The assumption of isentropic process is realistic, but the uniformity produce large error as the velocity must be a function of the

radius to keep the mass conservation. However, similar functionality (see boundary layer argument) is hopefully exist. In that case, the uniformity assumption produces smaller error than otherwise expected. Under this assumption the volume behind the shock has uniform pressure and temperature. This model is built under the assumption that there is no chemical reactions. For these assumptions, the mass can be expressed (for cylinder) as

$$m(t) = \frac{P\,V}{R\,T} \tag{6.78}$$

It can be noticed that all the variables are function of time with the exception of gas constant. The entering mass behind the shock is then

$$m_{in} = \overbrace{2\,\pi\,r}^{A}\,L\,U_y\,\rho_{inside} \tag{6.79}$$

The mass balance on the material behind the shock is

$$\dot{m}(t) - m_{in} = 0 \tag{6.80}$$

Substituting equations (6.78) and (6.79) into equation (6.80) results in

$$\frac{d}{dt}\frac{P\overbrace{\pi r^2 L}^{V}}{R\,T} - 2\pi r L U_x \rho_{outside} = 0 \tag{6.81}$$

or after simplification as

$$\frac{d}{dt}\frac{P\,r^2}{R\,T} - 2\,r\,U_x\,\rho_{outside} = 0 \tag{6.82}$$

The velocity M_x is given by equation (6.72) and can be used to expressed the velocity as

$$U_x = c_x\,M_x = \sqrt{k\,R\,T_{outside}}\,\overbrace{\sqrt{\left(\frac{k+1}{2\,k}\right)\left(\frac{P_{outside}}{P} - 1\right) + 1}}^{M_x} \tag{6.83}$$

Substituting equation (6.83) into equation (6.82) yields

$$\frac{d}{dt}\frac{P\,r^2}{R\,T} - 2\,r\,\sqrt{k\,R\,T_{outside}}\sqrt{\left(\frac{k+1}{2\,k}\right)\left(\frac{P_{outside}}{P} - 1\right) + 1} = 0 \tag{6.84}$$

which is a first order differential equation. The temperature behind the shock are affected by the conversion of the kinetic energy. The isentropic relationship for the radius behind the shock can be expressed as

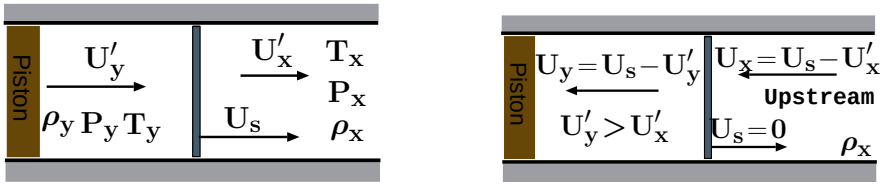
$$r = r_{ini}\left(\frac{P}{P_{ini}}\right)^{-\frac{1}{2\,k}} \tag{6.85}$$

Equations (6.84) and (6.85) can be substituted into (6.84) and denoting $\hat{P} = P/P_{ini}$ to yield

$$\frac{d}{dt}\frac{P_{ini}\,\hat{P}\,r_{ini}\hat{P}^{-\frac{1}{k}}}{R\,T_{ini}\hat{P}^{-\frac{k-1}{k}}} - 2\,r_{ini}\hat{P}^{-\frac{1}{2\,k}}\,\sqrt{k\,R\,T_{outside}}\,\sqrt{\left(\frac{k+1}{2\,k}\right)\left(\frac{\frac{P_{outside}}{P_{ini}}}{\hat{P}} - 1\right)} + 1 = 0 \tag{6.86}$$

6.3.4 Partially Open Valve

The previous section described a different category of the moving shock which can be generalized as one gas flows into another gas with a given lower velocity. The only limitation is that the "downstream' gas velocity must be higher than the "upstream" gas velocity as shown in Figure 6.19.



(a) Stationary coordinates (b) Moving coordinates

Fig. -6.17. A shock moves into a moving medium as a result of a sudden and complete open valve.

Here, the partially open value case is discussed when two kind of information are given. In the first case, $M_x{}'$ and $M_y{}'$ are given and rest of parameters have to be calculated. The second case deals with more realistic situation where the actual velocities and initial temperature is given.

Given Initial Mach Numbers

The relationship between the different Mach numbers on the "upstream" side is

$$M_x = M_{sx} - M_x{}' \tag{6.87}$$

The relationship between the different Mach on the "downstream" side is

$$M_y = M_{sy} - M_y{}' \tag{6.88}$$

An additional parameter has be supplied to solve the problem. A common problem is to find the moving shock velocity when the velocity "downstream" or the pressure is suddenly increased. It has to be mentioned that the temperature "downstream" is unknown (the flow of the gas with the higher velocity). The procedure for the calculations can be done by the following algorithm:

(a) Assume that $M_x = M_x^{'} + 1$.

(b) Calculate the Mach number M_y by utilizing the tables or Potto–GDC.

(c) Calculate the "downstream" shock Mach number $M_{sy} = M_y + M_y^{'}$

(d) Utilizing

$$M_x = \sqrt{\frac{T_y}{T_x}}\,(M_{sy}) - M_x^{'}$$

calculate the new "improved" M_x

(e) Check the new and improved M_x against the old one. If it is satisfactory, stop or return to stage (b).

Earlier, it was shown that the shock choking phenomenon occurs when the flow is running into a still medium. This phenomenon also occurs in the case where a faster flow is running into a slower fluid. The mathematics is cumbersome but results show that the shock choking phenomenon is still there (the Mach number is limited, not the actual flow). Figure 6.18 exhibits some "downstream" Mach numbers for various static Mach numbers, $M_y^{'}$, and for various static "upstream" Mach numbers, $M_x^{'}$. The figure demonstrates that the maximum can also occurs in the vicinity of the previous value (see following question/example).

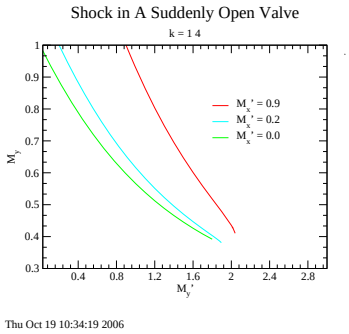


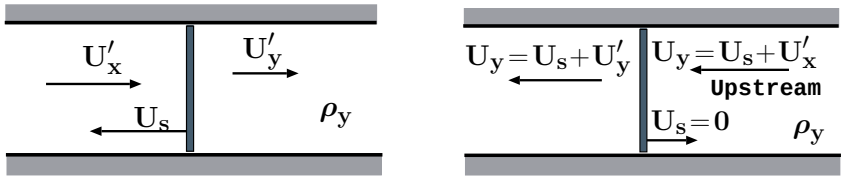
Fig. -6.18. The results of the partial opening of the valve.

Given Actual Velocities

In this case the algorithm is as the following:

1. Calculate the speed of sound upstream $c_x = \sqrt{k\,R\,T_x}$
2. Assume that solution for U_x is between c_x and $(U_y^{'} + c_x)$.
3. Calculate the Mach upstream number $M_x = U_x/c_x$
4. For guessed U_x (or M_x), calculate the downstream Mach number in moving coordinates M_y.
5. Calculate the velocity downstream the shock in moving coordinates $U_y = U_x(U_y/U_x)$
6. Compare the shock results from the right and left hand sides. $U_x + U_x^{'} \stackrel{?}{=} U_y + U_y^{'}$
7. Repeat the calculation and start at stage 2 of this algorithm with a new guess U_x.

6.3.5 Partially Closed Valve



(a) Stationary coordinates (b) Moving coordinates

Fig. -6.19. A shock as a result of a sudden and partially a valve closing or a narrowing the passage to the flow

The totally closed valve is a special case of a partially closed valve in which there is a sudden change and the resistance increases in the pipe. The information propagates upstream in the same way as before. Similar equations can be written (in the observer coordinates):

$$U_x = U_s + U_x{}' \tag{6.89}$$

The velocity at the observer coordinates (moving coordinates) is

$$U_y = U_s + U_y{}' \tag{6.90}$$

Thus the Mach numbers are

$$M_x = M_s + M_x{}' \tag{6.91}$$

$$M_y = M_s + M_y{}' \tag{6.92}$$

For given static Mach numbers the procedure for the calculation is as follows:

(a) Assume that $M_x = M_x{}' + 1$.

(b) . Calculate the Mach number M_y by utilizing the tables or Potto–GDC

(c) Calculate the "downstream" shock Mach number $M_{sy} = M_y - M_y{}'$

(d) Utilizing

$$M_x = \sqrt{\frac{T_y}{T_x}}\,(M_{sy}) + M_x{}'$$

calculate the new "improved" M_x

(e) Check the new and improved M_x against the old one. If it is satisfactory, stop or return to stage (b).

6.3.6 Worked–out Examples for Shock Dynamics

Example 6.4:
A shock is moving at a speed of 450 [m/sec] in a stagnated gas at pressure of 1$[Bar]$ and temperature of $27^{\circ}C$. Compute the pressure and the temperature behind the shock. Assume the specific heat ratio is k=1.3.

SOLUTION

It can be observed that the gas behind the shock is moving while the gas ahead of the shock is still. Thus, it is the case of a shock moving into still medium (suddenly opened valve case). First, the Mach velocity ahead of the shock has to calculated.

$$M_y{}' = \frac{U}{\sqrt{k\,R\,T}} = \frac{450}{\sqrt{1.3 \times 287 \times 300}} \sim 1.296$$

By utilizing Potto–GDC or Table (6.5) one can obtain the following table:

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.4179	0.50193	0.0	1.296	1.809	6.479	0.49695

The information about the iterations was obtained from Potto–GDC as

i	$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\mathbf{M_y}'$
0	1.0000	1.0000	1.0000	1.0000	1.2960
1	2.2960	0.51687	1.7217	5.8288	1.2960
2	2.3788	0.50652	1.7803	6.2661	1.2960
3	2.4050	0.50342	1.7992	6.4082	1.2960
4	2.4137	0.50242	1.8055	6.4552	1.2960
5	2.4165	0.50209	1.8076	6.4708	1.2960
6	2.4175	0.50198	1.8083	6.4760	1.2960
7	2.4178	0.50194	1.8085	6.4777	1.2960
8	2.4179	0.50193	1.8086	6.4783	1.2960
9	2.4179	0.50193	1.8086	6.4785	1.2960
10	2.4179	0.50193	1.8086	6.4785	1.2960
11	2.4179	0.50193	1.8086	6.4786	1.2960
12	2.4179	0.50193	1.8086	6.4786	1.2960

Using the above table, the temperature behind the shock is

$$T_y = T_y{}' = \frac{T_y}{T_x} T_x = 1.809 \times 300 \sim 542.7K$$

In same manner, it can be done for the pressure ratio as following

$$P_y = P_y{}' = \frac{P_y}{P_x} P_x = 6.479 \times 1.0 \sim 6.479[Bar]$$

The velocity behind the shock wave is obtained (for confirmation)

$$U_y{}' = M_y{}' c_y = 1.296 \times \sqrt{1.3 \times 287 \times 542.7} \sim 450 \left[\frac{m}{sec}\right]$$

End Solution

Example 6.5:
Gas flows in a tube with a velocity of $450[m/sec]$. *The static pressure at the tube is* $2Bar$ *and the (static) temperature of* $300K$. *The gas is brought into a complete stop by a sudden closing a valve. Calculate the velocity and the pressure behind the reflecting shock. The specific heat ratio can be assumed to be* $k = 1.4$.

SOLUTION

The first thing that needs to be done is to find the prime Mach number $M_x{}' = 1.2961$. Then, the prime properties can be found. At this stage the reflecting shock velocity is unknown.

Simply using the Potto–GDC provides for the temperature and velocity the following table:

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.0445	0.56995	1.2961	0.0	1.724	4.710	0.70009

If you insist on doing the steps yourself, find the upstream prime Mach, $M_x{}'$ to be 1.2961. Then using Table (6.3) you can find the proper M_x. If this detail is not sufficient then simply utilize the iterations procedure described earlier and obtain the following:

$\mathbf{i}$	$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\mathbf{M_y}'$
0	2.2961	0.53487	1.9432	0.0
1	2.042	0.57040	1.722	0.0
2	2.045	0.56994	1.724	0.0
3	2.044	0.56995	1.724	0.0
4	2.044	0.56995	1.724	0.0

The above table was obtained by utilizing Potto–GDC with the iteration request.

End Solution

Example 6.6:
What should be the prime Mach number (or the combination of the velocity with the temperature, for those who like an additional step) in order to double the temperature when the valve is suddenly and totally closed?

SOLUTION

The ratio can be obtained from Table (6.4). It can also be obtained from the stationary normal shock wave table. Potto-GDC provides for this temperature ratio the following table:

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.3574	0.52778	2.0000	3.1583	6.3166	0.55832

using the required $M_x = 2.3574$ in the moving shock table provides

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.3574	0.52778	0.78928	0.0	2.000	6.317	0.55830

End Solution

Example 6.7:
A gas flows in a pipe with a Mach number of 0.4. Calculate the speed of the shock when a valve is closed in such a way that the Mach number is reduced by half. Hint, this is the case of a partially closed valve case in which the ratio of the prime Mach number is half (the new parameter that is added in the general case).

SOLUTION

Refer to section (6.3.5) for the calculation procedure. Potto-GDC provides the solution of the above data

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.1220	0.89509	0.40000	0.20000	1.0789	1.3020	0.99813

If the information about the iterations is needed please refer to the following table.

i	M_x	M_y	$\frac{T_y}{T_x}$	$\frac{P_y}{P_x}$	$M_y{}'$
0	1.4000	0.73971	1.2547	2.1200	0.20000
1	1.0045	0.99548	1.0030	1.0106	0.20000
2	1.1967	0.84424	1.1259	1.5041	0.20000
3	1.0836	0.92479	1.0545	1.2032	0.20000
4	1.1443	0.87903	1.0930	1.3609	0.20000
5	1.1099	0.90416	1.0712	1.2705	0.20000
6	1.1288	0.89009	1.0832	1.3199	0.20000
7	1.1182	0.89789	1.0765	1.2922	0.20000
8	1.1241	0.89354	1.0802	1.3075	0.20000
9	1.1208	0.89595	1.0782	1.2989	0.20000
10	1.1226	0.89461	1.0793	1.3037	0.20000
11	1.1216	0.89536	1.0787	1.3011	0.20000
12	1.1222	0.89494	1.0790	1.3025	0.20000
13	1.1219	0.89517	1.0788	1.3017	0.20000
14	1.1221	0.89504	1.0789	1.3022	0.20000
15	1.1220	0.89512	1.0789	1.3019	0.20000
16	1.1220	0.89508	1.0789	1.3020	0.20000
17	1.1220	0.89510	1.0789	1.3020	0.20000
18	1.1220	0.89509	1.0789	1.3020	0.20000
19	1.1220	0.89509	1.0789	1.3020	0.20000
20	1.1220	0.89509	1.0789	1.3020	0.20000
21	1.1220	0.89509	1.0789	1.3020	0.20000
22	1.1220	0.89509	1.0789	1.3020	0.20000

End Solution

Example 6.8:
A piston is pushing air at 10m/sec. The air temperature was found to be 27°C. At certain time the piston velocity jump to 20 m/sec. Calculate the shock velocity and temperature behind the shock. Assume that $k = 1.4$ and $R = 287\, j/kg/K$.

SOLUTION

The shock velocity is between the speed of sound (upstream speed of sound that is before the shock) and speed of sound plus the new velocity of the piston. Utilizing the algorithm outline in section 6.3.4. The starting point for $U_x = \sqrt{1.4 \times 287 \times 300} = 347.2\, m/sec$ and the other side for $U_x = 347.2 + 20$.

i	$\mathbf{U_x}$	$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{U_y}}{\mathbf{U_x}}$	$\mathbf{U_{sx}}$	$\mathbf{U_{sy}}$
1	347.2	1.0	1.0	1.0	367.18	357.18
2	367.18	1.057	0.946	1.096	354.76	377.18
3	353.35	1.017	0.982	1.029	363.16	363.35
4	353.24	1.017	0.982	1.029	363.23	363.24
..	..	..	..	..	..	..
10	353.24	1.017	0.982	1.029	363.22	363.24

with the final results of

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.0174	0.98296	0.02880	0.05695	1.0116	1.0410	0.99999

End Solution

Example 6.9:

A piston is pushing air that flows in a tube with a Mach number of $M = 0.4$ and 300°C. The piston is accelerated very rapidly and the air adjoined the piston obtains Mach number $M = 0.8$. Calculate the velocity of the shock created by the piston in the air. Calculate the time it takes for the shock to reach the end of the tube of $1.0m$ length. Assume that there is no friction and the Fanno flow model is not applicable.

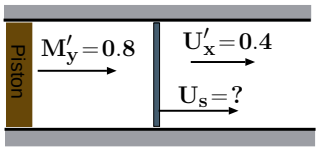


Fig. -6.20. Schematic of a piston pushing air in a tube.

SOLUTION

Using the procedure described in this section, the solution is

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.2380	0.81942	0.50000	0.80000	1.1519	1.6215	0.98860

The complete iteration is provided below.

$\mathbf{i}$	$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\mathbf{M_y}'$
0	1.5000	0.70109	1.3202	2.4583	0.80000
1	1.2248	0.82716	1.1435	1.5834	0.80000
2	1.2400	0.81829	1.1531	1.6273	0.80000
3	1.2378	0.81958	1.1517	1.6207	0.80000
4	1.2381	0.81940	1.1519	1.6217	0.80000
5	1.2380	0.81943	1.1519	1.6215	0.80000
6	1.2380	0.81942	1.1519	1.6216	0.80000

The time it takes for the shock to reach the end of the cylinder is

$$t = \frac{length}{\underbrace{U_s}_{c_x(M_x - M_x{}')}} = \frac{1}{\sqrt{1.4 \times 287 \times 300}(1.2380 - 0.4)} = 0.0034[sec]$$

End Solution

Example 6.10:
From the previous example (6.9) *calculate the velocity difference between initial piston velocity and final piston velocity.*

beginlatexonly
SOLUTION

The stationary difference between the two sides of the shock is:

$$\Delta U = U_y{}' - U_x{}' = c_y U_y{}' - c_x U_x{}'$$

$$= \sqrt{1.4 \times 287 \times 300}\left(0.8 \times \overbrace{\sqrt{1.1519}}^{\sqrt{\frac{T_y}{T_x}}} - 0.5\right)$$

$$\sim 124.4[m/sec]$$

End Solution

Example 6.11:

An engine is designed so that two pistons are moving toward each other (see Figure 6.21). The air between the pistons is at $1[Bar]$ *and* $300K$*. The distance between the two pistons is* $1[m]$*. Calculate the time it will take for the two shocks to collide.*

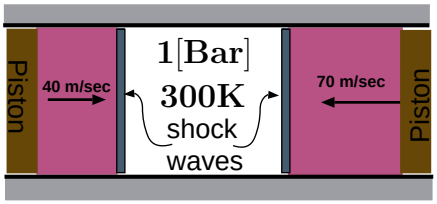


Fig. -6.21. Figure for Example 6.11.

SOLUTION

This situation is an open valve case where the prime information is given. The solution is given by equation (6.66), and, it is the explicit analytical solution. For this case the following table can easily be obtain from Potto–GDC for the left piston

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$	$\mathbf{U_y}'$	$\mathbf{c_x}$
1.0715	0.93471	0.0	0.95890	1.047	1.173	0.99959	40.0	347.

while the velocity of the right piston is

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$	$\mathbf{U_y}'$	$\mathbf{c_x}$
1.1283	0.89048	0.0	0.93451	1.083	1.318	0.99785	70.0	347.

The time for the shocks to collide is

$$t = \frac{length}{U_{sx1} + U_{sx2}} = \frac{1[m]}{(1.0715 + 1.1283)347.} \sim 0.0013[sec]$$

End Solution

6.4 Shock Tube

The shock tube has been a study tool with very little practical purposes. It have been used in many cases to examine certain phenomena because the high temperatures and pressure levels can be attained. Other situations can be understood and or extended from these phenomena by other tools. In the past most applications were related to military purposes. Now in these days there is a large role to civilian applications from a shock tube like configuration. Examples to such civilian application is rapid heating (for printing) and recycling of energy. A shock tube is made of a cylinder (pipe) with two chambers connected by a diaphragm. On one side the pressure is high, while the pressure on the other side is low (sometime approaching vacuum). The gas from the high

pressure section flows into the low pressure section when the diaphragm is ruptured[9]. A shock is created and travels to the low pressure chamber. This phenomenon is similar to the suddenly opened valve case described previously. At the back of the shock, the expansion waves reduce of pressure in the driver section. The temperature is known to reach several thousands degrees in a very brief period of time. The high pressure chamber is referred to in the literature is the *driver section and the low section is referred to as the* expansion section or the driven section.

This analysis is referred to ruptured diaphragm unless specifically referred to moving diaphragm. Initially, the gas from the driver section is coalescing from small shock waves into a large shock wave. In this analysis, it is assumed that this time is essentially zero. Zone 1 is an undisturbed gas and zone 2 is an area where the shock already passed. The assumption is that the shock is very sharp with zero width. On the other side, the expansion waves are moving into the high pressure chamber i.e. the driver section. The shock is moving at a supersonic speed (it depends on the definition, i.e., what reference temperature is being used and the medium behind the shock is also moving but at a velocity, U_3, which can be supersonic or subsonic in stationary coordinates. The velocities in the expansion chamber vary between three (five if the two non–moving zone are included) zones. In zone 3 refers to the original material that was at the high pressure chamber but is now at the same pressure as zone 2. The temperature and entropy at zone 3 is different from zone 2. Zone 4 is where the gradual transition occurs between the original high pressure to the low pressure. The boundaries of zone 4 are defined by initial conditions. The expansion front is moving at the local speed of sound in the high pressure section. The expansion back front is moving at the local speed of sound velocity but the actual gas is moving in the opposite direction in U_2. In the expansion chamber, the fronts are moving to the left while the actual flow of the gas is moving to the right (refer to Figure 6.22). In zone 5, the velocity is zero and the pressure is in its original value.

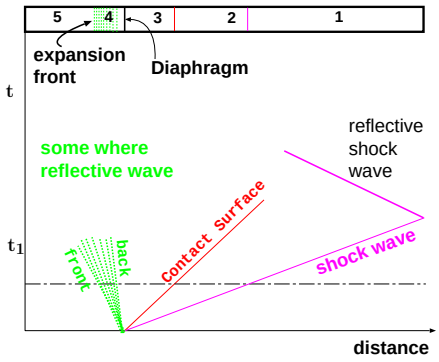


Fig. -6.22. The shock tube schematic with a pressure "diagram."

The properties in the different zones have different relationships. The relationship between zone 1 and zone 2 is that of a moving shock into still medium (again, this is a case of sudden opened valve). The material in zone 2 and 3 is moving at the same velocity (speed) but the temperature and the entropy are different, while the pressure in the two zones are the same. The pressure, the temperature and other properties in zone 4 aren't constant but continuous between the conditions in zone 3 to the conditions in zone 5. The expansion front wave velocity is larger than the velocity at the back front expansion wave velocity. Zone 4 is expanding during the initial stage (until the

[9]Now, a new design include the configuration of the moving diaphragm verse the ruptured is added.

expansion reaches the wall).

The shock tube is a relatively small length $1-2[m]$ and the typical velocity is in the range of the speed of sound, $c \sim \sqrt{340}$ thus the whole process takes only a few milliseconds or less. Thus, these kinds of experiments require fast recording devices (a relatively fast camera and fast data acquisition devices.). A typical design problem of a shock tube is finding the pressure to achieve the desired temperature or Mach number. The relationship between the different properties was discussed earlier and because it is a common problem, a review of the material is provided thus far.

The following equations were developed earlier and are repeated here for clarification. The pressure ratio between the two sides of the shock is

$$\frac{P_2}{P_1} = \frac{k_1 - 1}{k_1 + 1}\left(\frac{2\,k_1}{k_1 - 1}{M_{s1}}^2 - 1\right) \tag{6.93}$$

where k_1 the specific heat ratio in the expansion section (if two different gases are used). Rearranging equation (6.93) becomes

$$M_{s1} = \sqrt{\frac{k_1 - 1}{2\,k_1} + \frac{k_1 + 1}{2\,k_1}\frac{P_2}{P_1}} \tag{6.94}$$

Where M_{s1} is the front between the boundaries of zone 1 and 2. The velocity of this front can be expressed as

$$U_s = M_{s1} c_1 = c_1 \sqrt{\frac{k_1 - 1}{2\,k_1} + \frac{k_1 + 1}{2\,k_1}\frac{P_2}{P_1}} \tag{6.95}$$

The mass conservation $\rho_1 U_1 = \rho_2 U_2$ determines the relationship between the velocity as a function of the density ratio. The density ratio, using Rankine–Hugoniot relationship (6.26), can be expressed as a function of the pressure ratio as

$$\frac{U_1}{U_2} = \frac{\rho_2}{\rho_1} = \frac{1 + \left(\dfrac{k_1 + 1}{k_1 - 1}\right)\dfrac{P_2}{P_1}}{\dfrac{k_1 + 1}{k_1 - 1} + \dfrac{P_2}{P_1}} \tag{6.96}$$

The velocity in zone 2 in the moving coordinates relative to the shock is

$$U_2{}' = U_s - U_2 = U_s\left(1 - \frac{U_2}{U_s}\right) \tag{6.97}$$

Notice that U_s is equal to U_1. From the mass conservation, it follows that

$$\frac{U_2}{U_s} = \frac{\rho_1}{\rho_2} \tag{6.98}$$

$$U_2{}' = \overbrace{c_1 \sqrt{\frac{k_1-1}{2\,k_1} + \frac{k_1+1}{2\,k_1}\frac{P_2}{P_1}}}^{U_1} \left(1 - \frac{\overbrace{\frac{k_1+1}{k_1-1} + \frac{P_2}{P_1}}^{U_2/U_1}}{1 + \left(\frac{k_1+1}{k_1-1}\right)\frac{P_2}{P_1}} \right) \tag{6.99}$$

After rearranging equation (6.99) the result is

$$U_2{}' = \frac{c_1}{k_1}\left(\frac{P_2}{P_1} - 1\right) \sqrt{\frac{\frac{2\,k_1}{k_1+1}}{\frac{P_2}{P_1} + \frac{k_1-1}{1+k_1}}} \tag{6.100}$$

On the isentropic side, in zone 4, the flow is isentropic and disturbance is moving to the at the local speed of sound. Taking the derivative of the continuity equation, $d(\rho U) = 0$, and dividing by the continuity equation by $U\,\rho$ the following is obtained:

$$\frac{d\rho}{\rho} = -\frac{dU}{c} \tag{6.101}$$

Notice that the velocity, U was replaced with the sonic velocity (isentropic disturbance). Since the process in zone 4 is isentropic, applying the isentropic relationship ($T \propto \rho^{k-1}$) yields

$$\frac{c}{c_5} = \sqrt{\frac{T}{T_5}} = \left(\frac{T}{T_5}\right)^{\frac{1}{2}} = \left(\frac{\rho}{\rho_5}\right)^{\left(\frac{k_5-1}{2}\right)} \tag{6.102}$$

From equation (6.101) it follows that

$$dU = -c\frac{d\rho}{\rho} = c_5\left(\frac{\rho}{\rho_5}\right)^{\left(\frac{k_5-1}{2}\right)} d\rho \tag{6.103}$$

Equation (6.103) can be integrated as follows:

$$\int_{U_5=0}^{U_3} dU = \int_{\rho_5}^{\rho_3} c_5\left(\frac{\rho}{\rho_5}\right)^{\left(\frac{k_5-1}{2}\right)} d\rho \tag{6.104}$$

The results of the integration are

$$U_3 = \frac{2\,c_5}{k_5-1}\left(1 - \left(\frac{\rho_3}{\rho_5}\right)^{\frac{k_5-1}{2}}\right) \tag{6.105}$$

Or in terms of the pressure ratio as

$$U_3 = \frac{2\,c_5}{k_5 - 1}\left(1 - \left(\frac{P_3}{P_5}\right)^{\frac{k_5-1}{2\,k_5}}\right) \tag{6.106}$$

As it was mentioned earlier, the velocity at points 2 and 3 are identical, hence equation (6.106) and equation (6.100) can be combined to yield

$$\frac{2\,c_5}{k_5 - 1}\left(1 - \left(\frac{P_3}{P_5}\right)^{\frac{k_5-1}{2\,k_5}}\right) = \frac{c_1}{k_1}\left(\frac{P_2}{P_1} - 1\right)\sqrt{\frac{\frac{2\,k_1}{k_1+1}}{\frac{P_2}{P_1} + \frac{k_1-1}{1+k_1}}} \tag{6.107}$$

After some rearrangement, equation (6.107) is transformed into

$$\frac{P_5}{P_1} = \frac{P_2}{P_1}\left(1 - \frac{(k_5 - 1)\left(\frac{c_1}{c_5}\right)\left(\frac{P_2}{P_1} - 1\right)}{\sqrt{2\,k_1}\sqrt{2\,k_1 + (k_1+1)\left(\frac{P_2}{P_1} - 1\right)}}\right)^{-\frac{2\,k_5}{k_5-1}} \tag{6.108}$$

Or in terms of the Mach number, M_{s1}

$$\frac{P_5}{P_1} = \frac{k_1 - 1}{k_1 + 1}\left(\frac{2\,k_1 {M_{s1}}^2}{k_1 - 1} - 1\right)\left[1 - \frac{\left(\frac{k_1-1}{k_1+1}\right)\frac{c_1}{c_5}\left({M_{s1}}^2 - 1\right)}{M_{s1}}\right]^{-\frac{2\,k_5}{k_5-1}} \tag{6.109}$$

Using the Rankine–Hugoniot relationship (relationship across shock wave equation (6.25)) and the perfect gas model, the following is obtained:

$$\frac{T_2}{T_1} = \frac{\frac{k_1-1}{k_1+1} + \left(\frac{P_2}{P_1}\right)}{1 + \frac{k_1-1}{k_1+1}\left(\frac{P_2}{P_1}\right)} \tag{6.110}$$

By utilizing the isentropic relationship for zone 3 to 5 (and $P_2 = P_3$) results in

$$\frac{T_3}{T_5} = \left(\frac{P_3}{P_5}\right)^{\frac{k_5-1}{k_5}} = \left(\frac{P_2}{P_1} \Big/ \frac{P_5}{P_1}\right)^{\frac{k_5-1}{k_5}} \tag{6.111}$$

Solution of equation (6.109) requires that

$$0 < 1 - \frac{\left(\frac{k_1-1}{k_1+1}\right)\frac{c_1}{c_5}\left({M_{s1}}^2 - 1\right)}{M_{s1}} \tag{6.112}$$

Thus the upper limit of M_{s1} is determine by equation (6.112) to be

$$\left(\frac{k_1-1}{k_1+1}\right)\frac{c_1}{c_5}\left({M_{s1}}^2-1\right) > M_{s1} \tag{6.113}$$

The two solutions for the upper limit for M_{s1} are

$$M_{s1} = -\frac{\sqrt{\frac{\left({k_1}^2+2\,k_1+1\right)\,k_5\,R_5\,T_5+\left(4\,{k_1}^3-8\,{k_1}^2+4\,k_1\right)\,R_1\,T_1}{k_5\,R_5\,T_5}}-k_1-1}{\left(2\,k_1-2\right)\sqrt{\frac{k_1}{k_5}}\sqrt{\frac{R_1}{R_5}}\sqrt{\frac{T_1}{T_5}}} \tag{6.114}$$

and

$$M_{s1} = -\frac{\sqrt{\frac{\left({k_1}^2+2\,k_1+1\right)\,k_5\,R_5\,T_5+\left(4\,{k_1}^3-8\,{k_1}^2+4\,k_1\right)\,R_1\,T_1}{k_5\,R_5\,T_5}}+k_1+1}{\left(2\,k_1-2\right)\sqrt{\frac{k_1}{k_5}}\sqrt{\frac{R_1}{R_5}}\sqrt{\frac{T_1}{T_5}}} \tag{6.115}$$

The first limit equation (6.114) represents the after shock. While the second equation (6.115) represents the actual shock that occurs. The speed of sound on both sides affects the maximum Mach numbers. The typical value for air–air (under the assumption of constant air properties) is around 6. The lower limit of this maximum is around 2 for gas with lower specific heat. For a wide range this value can be assume to be between 4 to 8. For the case where the R and temperature value is plotted in Figure 6.23. When the temperature and the same gases are used (or the gases have the same R and the same k) the following is true

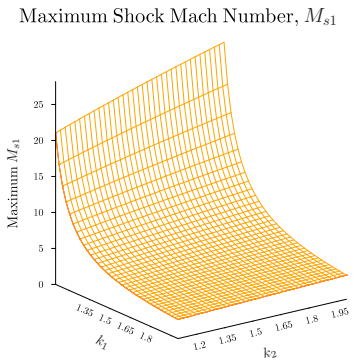


Fig. -6.23. Maximum Mach number that can be obtained for given specific heats.

$$M_{s1} = \frac{\sqrt{5\,{k_1}^2-6\,k_1+5}+k_1+1}{2\,k_1-2} \tag{6.116}$$

The physical significant is that when specific heat is approaching one (1) the material is more rigid and hence the information pass faster.

Example 6.12:
A shock tube with an initial pressure ratio of $\frac{P_5}{P_1} = 20$ and an initial temperature of $300K$. Assume that specific heat for both gases is equal to 1.4. Calculate the shock velocity and temperature behind the shock. If the pressure ratio is increased to $\frac{P_5}{P_1} = 40$ and the initial temperatures remain the same, would the temperature at point 3 (see Figure 6.22) increase or decrease? What is the reason for the change of the temperature? Assume that $R = 287\, j/kg/K$

SOLUTION

With the given pressure ratio of $\frac{P_5}{P_1} = 20$, equation (6.109) can be solved by iterations or any other numerical methods. The solution of the equation yields that $M_{s1} = 1.827$. This can be obtained using the half method where the solution start with limits of $1. + \epsilon$ to the limit shown in equation (6.116) see table below.

Iteration	M_{s1}	$\frac{P_5}{P_1}$	Iteration	M_{s1}	$\frac{P_5}{P_1}$
1	1.0000001	1.0000047	2	5.8333	21914856802.0
3	3.416667	2313.79	4	2.208	62.293
5	1.604167	9.89	6	1.91	25.44
7	1.755	16.0	22	1.827	∼20.00

The

Pressure ratio after shock can be calculated using (6.93) to obtain $P_2/\,P_1 = 3.729$ or using the standard shock table for shock Table 6.2.

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.8270	0.61058	1.5519	2.4020	3.7276	0.80062

The conditions at point 3 can be obtained using equation (6.111) as

$$T_3 = T_5 \frac{T_3}{T_5} = \left(\frac{P_2}{P_1} \Big/ \frac{P_5}{P_1} \right)^{\frac{k_5 - 1}{k_5}} = 300 \times \left(\frac{3.7276}{20} \right)^{\frac{0.4}{1.4}} = 185.6^{\circ}\mathrm{C} \qquad \text{(6.XII.a)}$$

And the speed of sound at this point is

$$c_3 = \sqrt{k5\, R\, T_3} = \sqrt{1.4 \times 287 \times 185.6} \sim 273.1 \left[\frac{m}{sec} \right] \qquad \text{(6.XII.b)}$$

All these calculations can be done in one step using Potto-GDC as

$\frac{\mathbf{P_5}}{\mathbf{P_1}}$	$\mathbf{M_{s1}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$	$\frac{\mathbf{T_2}}{\mathbf{T_1}}$	$\frac{\mathbf{P_5}}{\mathbf{P_3}}$	$\frac{\mathbf{T_5}}{\mathbf{T_3}}$	$\mathbf{U_{s5}}$
20.0000	1.8273	3.7287	1.5521	5.3637	1.6159	273.12

For the same condition with pressure ratio of $P_5/P_1 = 40$ results in

$\frac{\mathbf{P_5}}{\mathbf{P_1}}$	$\mathbf{M_{s1}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$	$\frac{\mathbf{T_2}}{\mathbf{T_1}}$	$\frac{\mathbf{P_5}}{\mathbf{P_3}}$	$\frac{\mathbf{T_5}}{\mathbf{T_3}}$	$\mathbf{U_{s5}}$
40.0000	2.0576	4.7726	1.7348	8.3812	1.8357	256.25

As indicated from this table, the temperature in zone 3 became more cold in comparison with the pressure. It also can be observed that the shock Mach number is larger.

End Solution

As demonstrated by Example (6.12) the shock Mach number is much smaller than the actual small upper limit. This can be plotted as a function of the given pressure ratio and for various specific heat ratios. This plot is presented in Figure 6.24 for three combination of the specific heats.

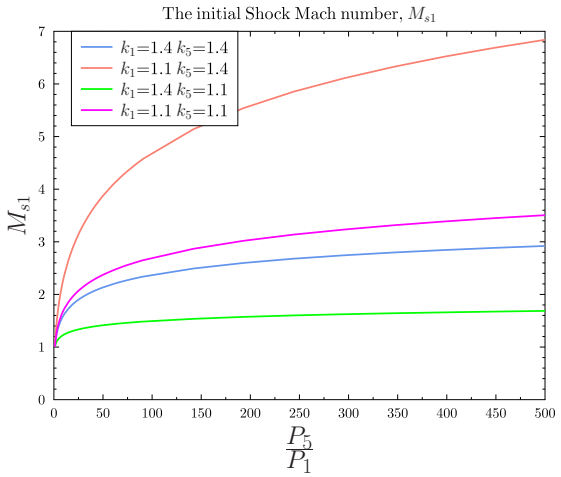


Fig. -6.24. The Mach number obtained for various parameters.

The Figure 6.24 shows that the maximum Mach is larger and lower for the mixed value of k as compared to when k_1 and k_5 are equal. When the pressure ratio, P_5/P_1 reaches to value larger than 50 the Mach number can be treat as constant for the specific combination.

Example 6.13:
A shock tube with the conditions given in Example 6.12 has a length of 4 meter and diaphragm exactly in the middle. Calculate the time it takes to the shock reach half

way to the end (1 meter from the end). Plot the pressure and temperature as a function of location for initial pressure ratio of 20, 40, 100, 500.

SOLUTION

The Mach number can be found in the method described in Example 6.12 and present in the following table[10].

$\frac{\mathbf{P_5}}{\mathbf{P_1}}$	$\mathbf{M_{s1}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$	$\frac{\mathbf{T_2}}{\mathbf{T_1}}$	$\frac{\mathbf{P_5}}{\mathbf{P_3}}$	$\frac{\mathbf{T_5}}{\mathbf{T_3}}$	$\mathbf{U_{s5}}$
20.00	1.8273	3.7287	1.5521	5.3638	1.6159	273.12
40.00	2.0576	4.7726	1.7348	8.3813	1.8357	256.25
100.00	2.3711	6.3922	2.0129	15.6440	2.1940	234.39
500.00	2.9215	9.7909	2.5878	51.0681	3.0764	197.94

The time can be obtained by

$$t = \frac{L}{U_{s1}} = \frac{0.5}{c_1 M_{s1}} = \frac{0.5}{\sqrt{k_1 R_1 T_1} M_{s1}} = \frac{.5}{\sqrt{1.4 \times 287 \times 300} M_{s1}} \qquad \text{(6.XIII.a)}$$

Where M_{s1} is given by the table above. For example for first case of $\frac{P_5}{P_1} = 20$

$$t = \frac{0.5}{\sqrt{1.4 \times 287 \times 300} \times 1.8273} \sim 0.00079\,[sec] \qquad \text{(6.XIII.b)}$$

The location of points 3 and point 4 are obtained by using the time obtained in in previous calculations times the local sound velocity. The velocity at boundary between zone 3 to zone 4 is

$$U_3 = \sqrt{k_5 R T_3} = \sqrt{\frac{k_5 R T_5}{\frac{T_5}{T_3}}} = \sqrt{\frac{1.4 \times 287 \times 300}{1.6159}} = 273.1 \left[\frac{m}{sec}\right] \qquad \text{(6.XIII.c)}$$

Thus the distance is

$$\ell_3 = c_3 t = 273.1 \times 0.00157 = 0.43[m] \quad \text{from the shock tube center} \qquad \text{(6.XIII.d)}$$

The location of border between zone 2 to 3 can be obtained using the same velocity (but to different direction) as

$$\ell_2 = c_2 t = c_3 t = 0.43[m] \qquad \text{(6.XIII.e)}$$

[10] This table was generated using Potto GDC. The new version was not released yet since the interface is under construction.

where the velocity of point 4 (between zone 4 and 5) is

$$U_4 = \sqrt{k_5\, R\, T_5} = \sqrt{1.4 \times 287 \times 300} \sim 347.2 \left[\frac{m}{sec}\right] \tag{6.XIII.f}$$

The distance of point 4 is

$$\ell_4 = c_3\, t = 347.2 \times 0.00157 = 0.545[m] \tag{6.XIII.g}$$

With the above information, the figure can be plotted with the exception of the zone 4, where the relation as a function of x have to be developed. Figure 6.25 schematic of the shock tube where blue is the region of the undisturbed gas and the purple color is the region where the gas went complete isentropic reduction and it is in its lowest temperature. The element shown in the Figure 6.25 is moving with velocity c_x. One moving with the element observed that the material is entering the differential element at the velocity c_x. Thus, mass conservation on the moving element reads

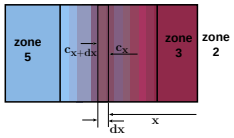


Fig. -6.25. Differential element to describe the isentropic pressure.

$$\rho\, A\, \frac{dx}{dt} = \rho\, U\, A = \rho\, c_x\, A \tag{6.XIII.h}$$

or

$$\frac{dx}{dt} = c_x = \sqrt{k\, R\, T_x} = \sqrt{k\, R\, T_5}\, \sqrt{\frac{T_x}{T_5}} = \sqrt{k\, R\, T_5}\, \left(\frac{P_x}{P_5}\right)^{\frac{k_5 - 1}{2\, k_5}} \tag{6.XIII.i}$$

equation (6.XIII.i) is a linear first order differential equation. It can be noticed the P_x is fix however the x is the variable and thus the solution is

$$x = \sqrt{k\, R\, T_5}\, \left(\frac{P_x}{P_5}\right)^{\frac{k_5 - 1}{2\, k_5}}\, t + C \tag{6.XIII.j}$$

At $t = 0$ the distance is zero. Thus the constant is zero and the solution is

$$x = \sqrt{k\, R\, T_5}\, \left(\frac{P_x}{P_5}\right)^{\frac{k_5 - 1}{2\, k_5}}\, t \tag{6.XIII.k}$$

This equation is applicable for the edge state which were already used. This equation simply states that velocity is relative to the pressure. This result is similar to method of characteristic which will be discussed later. For example, for the pressure ratio of

$\frac{0.5}{P_5/P_3}$ is

$$c_x = \sqrt{k\,R\,T_5}\left(\underbrace{\frac{\frac{0.5\,P_5}{P_3}}{2.6819}}\right)^{\frac{1-k_5}{2\,k_5}} t \tag{6.XIII.l}$$
$$= \sqrt{1.4 \times 287 \times 300} \times 2.6819^{\frac{1-1.4}{2.8}} \times 0.00157 = 301.6 \times 0.00157$$
$$\sim 0.47[m]$$

The same can be done for every point of the pressure range of $P_x/P_5 = 1$ to $P_x/P_5 = 1/5.3638 = 0.186$ or the temperature range.

End Solution

6.4.1 Special Shock Dynamics

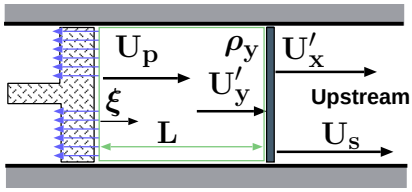


Fig. -6.26. Porous piston pushing gas and there is shock similar to open valve case.

In the previous Section 6.3 various standard cases of shock dynamics were presented. In this section, a discussion on irregular cases is presented. The partial open and partial close valve are the most general cases. However, the question or the given information can be different which depends on the physical conditions. For example, in the "classic" partial open valve the velocity or the Mach number is specified. In other cases, the flow rate is specified as a boundary condition. The solution process is more complicate in the class cases. Consider a porous piston or leaking piston pushing gas. In that case the piston velocity as boundary condition can be specified but rather the flow rate. To make this point clear consider the following situation. A flow is pushed by a piston made from a porous material as shown in Figure 6.26. As to previous cases, the condition(s) that can be specified on the flow at the piston head. The flow rate (through the piston) at the piston head is a function of the pressure at that point. Another point the pressure downstream from the shock is not uniform. The solution of problem requires Specify the boundary condition.

In this section, a limited discussion on the governing equations and boundary conditions is presented. The typical control volume in these case is made from the shock front and piston head (shown in the Figure 6.26 by the green box). The coordinates are either attached to the shock front or to piston head. The control volume is expanding with time. The velocity of the shock is denoted as U_s. While the velocity through the porous piston is not uniform, however,it is considered to be uniform for simplicity reasons. Using Darcy's law (in coordinates fixed to the piston under assumption of the constant piston velocity) for the piston reads

$$U_{out} = K_p\,(P_p - P_{out}) \tag{6.117}$$

where K_p represents the characteristic resistance and geometry of the piston. where p denotes the conditions at the piston head or in other words just before the gas flows into the piston. The boundary conditions at the shock interface are identical to stationary shock in coordinates moving with the shock. For the solution it is assumed that shock conditions are at steady state. Later these conditions can be improved. The mass conservation of the control volume is

$$\frac{d\,m}{dt} = m_{in} - m_{out} \tag{6.118}$$

The relationship between the velocities in the moving coordinates and the stationary coordinates are expressed as

$$U_y{}' = U_s - U_y \quad \text{or/and} \quad U_y = U_s - U_y{}' \quad \text{or/and} \quad U_p = U_s - U_p{}' \tag{6.119}$$

Where p denotes the conditions at the piston head or in other words just before the gas flows into the piston, and y denotes the downstream shock conditions. Note that $U_y{}'$ is measured to right as positive. In this case, the density in both coordinates is the same, $\rho_y = \rho_y'$.

The density after the downstream shock, ρ_y can be calculated based on the density on the other side (upstream) of the shock equation (6.24). It can be noticed that velocity in the control volume, $U_x = U_s$, is the shock velocity. The flow rate in the control volume (written in the moving coordinates) is

$$m_{in} = \rho_y \left(\overbrace{U_s - U_y{}'}^{U_y} \right) A \tag{6.120}$$

The flow rate out (at the piston) can be expressed as

$$m_{out} = \rho_p \, U_{out} \, A \tag{6.121}$$

The relationship between the density and pressure in side the control volume can be considered isentropic hence the relationship can be expressed as

$$\frac{\rho_p}{\rho_y} = \left(\frac{P_p}{P_y} \right)^{\frac{1}{k}} \tag{6.122}$$

Using the relationship for normal shock

$$\frac{\rho_x}{\rho_y} = \frac{1 + \left(\dfrac{k+1}{k-1} \right) \left(\dfrac{P_y}{P_x} \right)}{\left(\dfrac{k+1}{k-1} \right) + \left(\dfrac{P_y}{P_x} \right)} \tag{6.123}$$

When the x denotes the condition upstream of the shock which consider known.

The shock Mach number can be calculated utilizing several relationships. First, the pressure ratio P_r/P_x determines the Mach number as

$$M_s = M_x = \sqrt{\frac{k_1 - 1}{2\,k_1} + \frac{k_1 + 1}{2\,k_1}\frac{P_x}{P_y}} \tag{6.124}$$

The mach number after shock can be calculated using the normal shock relationship as

$${M_y}^2 = \frac{{M_x}^2 + \dfrac{2}{k-1}}{\dfrac{2\,k}{k-1}\,{M_x}^2 - 1} \tag{6.125}$$

The velocity can be calculated using the Mach number definition

$$U_y = M_y\,\sqrt{k\,R\,T_y} \tag{6.126}$$

The mass in the control volume (in the piston moving coordinate) is

$$m_{cv} = A\int_0^{L(t)} \rho(\xi) d\xi \tag{6.127}$$

The isentropic relationship ties the pressure at the downstream the shock, y ($\rho(\xi) = \rho_y(P(\xi)/P_y)^{1/k}$. The density profile is

$$\rho(\xi) = \frac{\rho_y}{{P_y}^{1/k}}\,f_p\left(\frac{\xi}{L}\right) \tag{6.128}$$

Hence,

$$m_{cv} = \frac{A\,\rho_y\,L}{{P_y}^{1/k}}\int_0^{\frac{L(t)}{L}} f_p\left(\overbrace{\frac{\xi}{L}}^{\eta}\right)\overbrace{\frac{d\xi}{L}}^{d\eta} \tag{6.129}$$

Where f_p is the function describing the distribution of the density.

Thus the derivative with respect to time of the mass in the control volume is

$$\frac{d\,m_{cv}}{dt} = \frac{d}{dt}\left(\frac{A\,\rho_r\,L}{{P_r}^{1/k}}\int_0^1 f_p\left(\eta\right)d\eta\right) \tag{6.130}$$

Since $L(t)$ is a function of time but the integral is not function of the time anymore. it can be written that

$$\frac{d\,m_{cv}}{dt} = \frac{A\,\rho_r}{{P_r}^{1/k}}\int_0^1 f_p\left(\eta\right)d\eta\,\overbrace{\frac{d\,L}{dt}}^{U_s} \tag{6.131}$$

The temperature, T_y (refer to Figure 6.1), can be found from the shock relationship as

$$T_y = T_x \left(\frac{P_y}{P_x}\right) \left(\frac{\frac{k+1}{k-1} + \frac{P_y}{P_x}}{1 + \frac{k+1}{k-1}\frac{P_y}{P_x}} \right) \tag{6.132}$$

The density at the piston area, ρ_p, is a function of the shock Mach number.

In this set of equations there are 7 unknowns and 7 equations which can be solved. The discussion for the numerical method of solution is not presented here at this stage.

6.4.2 Shock Tube Thermodynamics Considerations

The shock tube applications experiencing transition from mostly military to civilian utilization. The civilian utilization of the shock tube requires economizing the shock tube design. As appose to construction for military usage, the civilian applications require reduction of the cost. In industrial applications the cost of operation and efficiency plays an important role. One of the civilian application is that the shock tube possibly can be used for very rapid heating. An example of civilian application is the printing industry where a rapid heating is required. After spraying the ink, the ink has to be rapidly heated and cooled to be used by the final user. Today, this "baking" process is obtained by laser light which consumes large amount of energy and required additional energy for the cooling. Shock tubes also can be used to absorb energy to create high temperature difference which later can be used "recycle" the energy with a larger efficiency (according to Carnot cycle).

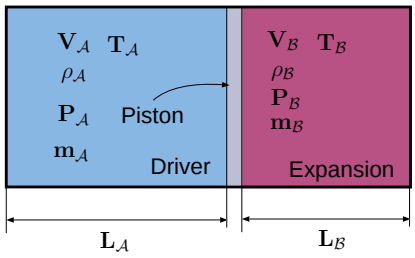


Fig. -6.27. Shock tube schematic for thermodynamics consideration at initial stage.

Examination of the maximum temperature potential of devices of a shock–tube–like–configuration leads to a new design requirement on the diaphragm. This requirement of the shock tube dictates that the diaphragm should be moving as a novel concept. The basic understanding of the process can be gained by looking at the underline thermodynamical principles which control the process. In Chapter 2 (page 33) the concept of the pressure potential was introduced. Here, this pressure potential concept will be extended.

In the previous case (see Section 2.1.1), the compressed chambered was working against vacuum. The general case refers to the case where the one chamber work or potential is done on the adjoin chamber. Figure 6.27 exhibits two chambers separated by a moving diaphragm (denoted as a piston in the Figure 6.27). The left chamber contains high pressure gas while the pressure on the right chamber is very low. The

maximum work is obtained when the driver chamber undergoes adiabatic isentropic expansion. In the same vain of argument, the expansion chamber has to undergo adiabatic compression however, the process does not have to be isentropic. The temperature (or the internal energy) in the driven chamber will be a function of the energy or work transferred into regardless the thermodynamical path.

At the end point of the process, after the piston (moving diaphragm) is allowed to move, an pressure equilibrium is obtained. This equilibrium means that the pressures at the two chambers is identical but the temperature is different. The equilibrium point depends on the several parameters. In the book "Basics of Fluid Mechanics" presents a discussion about the dimensional analysis of this topic (also see a youtube video by this author on this topic). The internal and external engines are constructed by utilizing the temperature difference (i.e. the internal combustion engine.)[11]. The shock tube is a device operates in the opposite direction where the pressure difference is used to increase the temperature difference. The reverse order was never analyzed thermodynamically and systematically until now. There are thermodynamical paths on which the pressure difference can yield a minimal temperature increase, if any. For example, two chambers with a large pressure difference can be connected via a large resistance pipe. The fluid from the high pressure chamber flows slowly to the second chamber resulting in a very minimal temperature change. The temperature does not necessarily increase. If the fluid is liquid, then a considerable temperature reduction is observed in both chambers (see the common refrigeration cycle).

The standard construction of ruptured diaphragm leads to analysis of maximum temperature in the past[12]. Perhaps the most famous of Zel'Dovich[13] ideas which can be summarized by the concept of large energy (large specific heat on the driver side) and low heat capacity in the expansion section. This analysis were based on the shock dynamics (i.e. successive shocks or sock in a vacuum). In this book it was shown that, regardless pressure ratio, the shock strength has a maximum in the shock tube configuration.

The standard analysis for the first shock's strength and velocity is done by using the one-dimensional approach. It can be understood that the mixingless situation will increase the maximum temperature. This analysis does not check what is the limit in the current configuration. This analysis evaluates the maximum temperature by looking at the potential (pressure and internal) energy and evades the successive shock waves analysis. In other words, this shock analysis is not addressed and is bypassed. Furthermore, the current analysis will also check, regardless to the configuration, what is the maximum temperature. Thus, conceivably this analysis could lead to a different configuration of an improved shock tube with a higher efficiency.

The introduction of the notion of the maximum temperature suggests the concept of the shock tube efficiency. This author is not familiar with any definition of the shock tube efficiency in the literature. There are several potential definitions which can

[11]Even the the absorption refrigerator cycle is using the temperature difference for cooling.

[12]This point is presented in this version because it so novel that most researchers in the area not familiar with these points and concepts. It is will be removed when the information will spread around.

[13]These proposed by Zel'Dovich's proposals were repeated by several other researchers.

be employed. The simplest definition can be referred to as the impossible maximum efficiency. This definition refers to the ratio of the actual temperature to the temperature if all the available potential pressure energy in the driver section was transferred to the expansion section. For a model without reactions, it can be written as

$$\eta_{imaginary} = \frac{\Delta E_{u_{\text{expansion}}}}{(P_{A1} - P_{B1})\, A\, L_{\mathcal{B}1}} \tag{6.133}$$

In writing equation (6.133) it is realized that the potential energy is the work done by the driver section from the initial condition (see Fig. 6.27) to the final pressure on the expansion section moving along the entire expansion section. The advantage of this definition is its simplicity; there is no need to find the final equilibrium pressure or the equilibrium location. The disadvantage of this method is that it does not provide a thermodynamical path on which the maximum temperature can be obtained.

To remedy the last problem, another definition, based on the driver section expansion to the point where driver section and the driven section are brought into equilibrium, is proposed as

$$\eta_{real} = \frac{\Delta E_{u_{\text{expansion}}}}{\int_{\mathcal{A}1}^{e} A\, P\, dx} \tag{6.134}$$

The final pressure is unknown and depends on the path this equilibrium is achieved. The gas expansion process in the driver section can move on an arbitrary path such as isothermal or isentropic. The final pressure and temperature are a function of the path process. The second law of thermodynamics prescribes that the maximum work which can be extracted during the process (in driver section) must be isentropic[14]. The isentropic path reduces the driver temperature to a minimum thus a maximum energy extraction is achieved. The temperature increase of the expansion (driver) section is achieved by the compression done by the driver section when the internal energy and the pressure potential provide the energy. The isentropic process is achieved when the flow is very slow[15] and hence the equilibrium will be achieved without oscillations. The work is transformed and the expansion section increases the gas internal energy. Note that the exchange or transfer of work between the two sides causes the reduction of the driver section temperature. The maximum temperature that will be obtained when all the compression potential is converted to internal energy and the velocity is zero in the system. That is, the pressure at the two sides is equal with no further movement of the gas.

[14] The point of this excesses is to extract maximum work. Then the heat addition (either from reactions or transferred) will increase the potential work. This point is not discussed in this work and could be extended see example later.

[15] The actual process in the driver section is very fast. Yet, in the initial part when the first shock and the reflecting wave take place, the sound waves in the driver section are assumed to be isentropic (as sound wave assumed to be shock with zero amplitude and hence assumed to be isentropic). This assumption is very common to any shock tube analysis tube regardless of the fact that sound wave are quite fast. The statement is directed to the stage after these initial shocks to the point when the interface moves slowly to the equilibrium. This discussion does not address how to achieve this process technically.

The starting point of the process, the driver chamber is with high uniform pressure while the driven section is at low pressure. At the end point, shown in Figure 6.28, the pressure driver section is at equilibrium with the pressure of the deriven section. The chamber on the left (driver section) is denoted $\mathcal{A}$ and the right (driven section) is denoted as $\mathcal{B}$. The temperature, pressure, and volume are denoted as 1 for the initial state and e or 2[16] for the final state. The gas expands from the left (high pressure) to the right (low pressure).

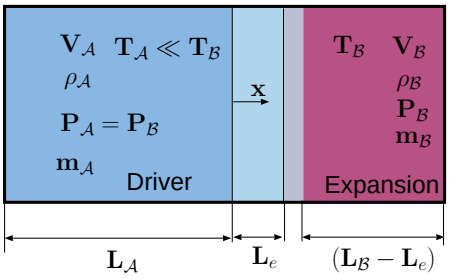


Fig. -6.28. The final or equilibrium stage in the shock tube.

At the final stage, the gas remains at equilibrium and does not mix (moving diaphragm case). The mixing process takes time and the characteristic time is larger. Further, in most cases, the mixing process reduces the maximum temperature. While there might be some gases combinations which increase the maximum temperature, these cases are not investigated in this analysis. Hence, the mixing process element is neglected.

The final gas pressure can be obtained because the path is known (adiabatic isentropic process). Utilizing the ideal gas model, the mass at both chamber remains constant and equal to

$$m_{ji} = \frac{P_{ji}\, V_{ji}}{R_j\, T_{ji}} \tag{6.135}$$

Where i indicates the state (initial, final, intermediate, equilibrium) and j indicates if it is in $\mathcal{A}$ or $\mathcal{B}$. The density, ρ, as a function of the distance, x, can be expressed as

$$\rho_j(x) = \frac{P_{j1}\, L_{j1}}{R_j\, T_{j1}\, (L_{j1} \pm x)} \tag{6.136}$$

Where the $\pm$ sign is plus for the driver and minus for the expansion section. Notice that in equation (6.136), the volumes were substituted by the lengths since the cross section is uniform.

Utilizing the ideal gas model for the relationship between the initial condition and any intermediate point on the path, the following can be written as

$$P_j(x) = P_{j1}\, \frac{T_j(x)\, L_{j1}}{T_{j1}\, (L_{j1} \pm x)} \tag{6.137}$$

Using the isentropic relationship for the internal process for the driver section, the following can be written

$$P_{\mathcal{A}}(x) = P_{\mathcal{A}1}\, \left(\frac{L_{\mathcal{A}1}}{L_{\mathcal{A}1} + x} \right)^k \tag{6.138}$$

[16] In this case, the notation or the states of e and 2 are identical. In other situations, the equilibrium and the final states are different.

Introducing a dimensionless variable

$$\xi = \frac{x}{L_{\mathcal{A}} + L_{\mathcal{B}}} = \frac{x}{L} \tag{6.139}$$

where $L = L_{\mathcal{A}} + L_{\mathcal{B}}$ and which implicitly indicates that with the notation used herein as $\xi_{\mathcal{A}} = L_{\mathcal{A}1}/L$, $\xi_{\mathcal{B}} = L_{\mathcal{B}1}/L$, and $\xi_e = L_e/L$.

The work done by the driver section is

$$\frac{W_{\mathcal{A}1 \longrightarrow e}}{A} = \int_0^{L_e} P_{\mathcal{A}1} \left(\frac{L_{\mathcal{A}1}}{(L_{\mathcal{A}1} + x)} \right)^{k_{\mathcal{A}}} dx \tag{6.140}$$

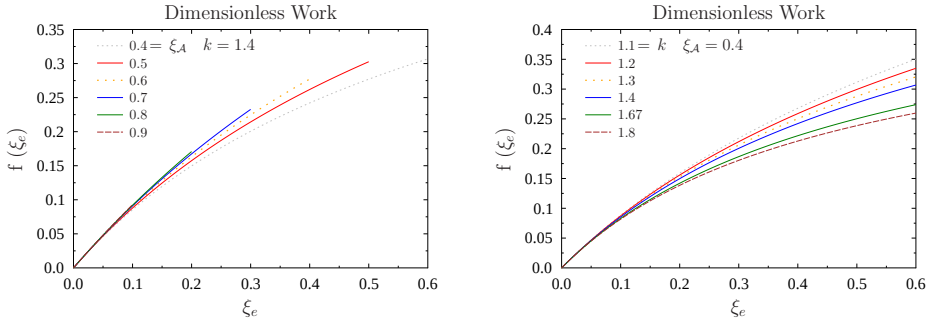
Equation (6.140) can be transformed into a dimensionless form as

$$\frac{W_{\mathcal{A}}}{A\, L\, P_{\mathcal{A}1}} = \int_0^{\xi_e} \left(\frac{\xi_{\mathcal{A}}}{\xi_{\mathcal{A}} + \xi} \right)^{k_{\mathcal{A}}} d\xi \tag{6.141}$$

Carrying the integration of equation (6.141) yields

$$\frac{W_{\mathcal{A}}}{A\, L\, P_{\mathcal{A}1}} = \frac{\xi_{\mathcal{A}} \left(\xi_{\mathcal{A}} + \xi_e\right)^k - \left(\xi_{\mathcal{A}} + \xi_e\right) \xi_{\mathcal{A}}{}^k}{(k-1)\, \left(\xi_{\mathcal{A}} + \xi_e\right)^k} \overset{def}{\equiv} f\left(\xi_e\right) \tag{6.142}$$

In the case where the ideal gas model cannot be used, the left hand side has to be replaced with the integral representing the internal energy change. When the specific heat, C_v, is significantly varied with the temperature in the range the common value has to be replaced with the averaged specific heat, $\overline{C_v}$.



(a) Dimensionless work as a function for various ξ_e, $k = 1.4$

(b) Dimensionless work as a function for various k, $\xi_e = 0.4$

Fig. -6.29. Dimensionless work as a function of dimensionless length of the expansion for the shock tube.

The work is done by the driver section as a function of ξ_e and is depicted in Figure 6.29. It must be alluded that the work value can be approximated as a simple, ξ_e almost up to 0.1 (and could be used in the solution phase).

Example 6.14:
A shock tube made from a pipe with inside radius of $0.1[m]$ *separated into sections by a diaphragm. The length ratio of the driver section to deriven section is 0.4. The initial pressure at the driver section is 1000 [Bar] and its length is 2 [m]. Calculate the energy available in the driver section.*

SOLUTION

The calculations require that the ξ_e will be found. This information will be available at a later stage however now, as first approximation $\xi_{\mathcal{A}}$ can be used instead. Thus, the ratio of the section lengths, $L_{\mathcal{A}}/L_{\mathcal{B}}$, has to converted into $\xi_{\mathcal{A}}$. Utilizing equation (6.139)

$$\frac{1}{\xi_{\mathcal{A}}} = \frac{L_{\mathcal{A}} + L_{\mathcal{B}}}{L_{\mathcal{A}}} = 1 + \frac{L_{\mathcal{B}}}{L_{\mathcal{A}}} \longrightarrow \xi_{\mathcal{A}} = \frac{1}{1 + \dfrac{L_{\mathcal{B}}}{L_{\mathcal{A}}}} \tag{6.XIV.a}$$

$$\xi_{\mathcal{A}} = \frac{1}{1 + \dfrac{1}{0.4}} \sim 0.285$$

As estimate according the Figure 6.29 the dimensionless work $f(\xi_e) \sim f(\xi_{\mathcal{A}}) \sim 0.25$ Thus the work will be

$$W = A\,L\,P_{\mathcal{A}1} f(\xi) = \pi \times 0.1^2 \times \overbrace{2\left(1 + \frac{1}{0.4}\right)}^{L} \times 1000\,10^5 \times 0.25 \sim 2748.893kJ \tag{6.XIV.b}$$

This number is only the estimate and the exact number will be calculated later when dimensionless length, ξ_e is calculated.

End Solution

Energy balance on the driver section reads

$$E_{u_{\mathcal{A}1}} - E_{u_{\mathcal{A}e}} = W_{\mathcal{A}1 \longrightarrow e} = f\left(\xi_e\right)\, A\,L\,P_{\mathcal{A}1} \tag{6.143}$$

Utilizing the expression for the work (equation (6.142)) and further assuming an ideal gas model with a constant specific heat equation (6.143) can be written as

$$\frac{m_{\mathcal{A}}\, C_{v\,\mathcal{A}}\, T_{\mathcal{A}1}\left(1 - \dfrac{T_{\mathcal{A}e}}{T_{\mathcal{A}1}}\right)}{A\,L\,P_{\mathcal{A}1}} = f\left(\xi_e\right) \tag{6.144}$$

Substituting the isentropic relationship between the temperature and pressure, and equation 6.135 into equation (6.144) yields

$$\frac{\xi_{\mathcal{A}}\, C_{v\,\mathcal{A}}}{R_{\mathcal{A}}}\left(1 - \left(\frac{P_{\mathcal{A}e}}{P_{\mathcal{A}}1}\right)^{\frac{k_{\mathcal{A}}-1}{k_{\mathcal{A}}}}\right) = f\left(\xi_e\right) \tag{6.145}$$

Equation (6.145) puts a cap or a limit on the driver pressure ratio. The right hand side is limited by ξ_e which is limited by $\xi_{\mathcal{B}}$. Substituting this value into equation (6.145) yields

$$P_{\mathcal{A}e}|_{cap} = P_{\mathcal{A}1}\left(1 - \frac{R_{\mathcal{A}}\, f(\xi_{\mathcal{B}})}{C_{v\,\mathcal{A}}\, \xi_{\mathcal{A}}}\right)^{\frac{k_{\mathcal{A}}}{k_{\mathcal{A}}-1}} \tag{6.146}$$

This is the lowest equilibrium pressure that can be achieved on the isentropic path. It can be noticed that ratio $R_{\mathcal{A}}\, f(\xi_{\mathcal{B}}) < C_{v\,\mathcal{A}}\, \xi_{\mathcal{A}}$ is always true. Hence, this indicates that the maximum energy which can be extracted from the driver depends on the ratio of the driver length to the driven section length. This limit means that any further increase of the pressure leads to a further increase in the equilibrium pressure and hence, the potential maximum energy extraction[17].

The energy balance[18] of the driven section (compressed gas) is

$$E_{u\mathcal{B}1} - E_{u\mathcal{B}e} = -W_{\mathcal{B}1\longrightarrow e} = -f(\xi_e) \tag{6.147}$$

Utilizing the expression for the work (equation (6.142) and further assuming ideal gas model with a constant specific heat, equation (6.147) can be written as

$$\frac{m_{\mathcal{B}}\, C_{v\mathcal{B}}\, (T_{\mathcal{B}e} - T_{\mathcal{B}1})}{A\, L\, P_{\mathcal{A}1}} = -f(\xi_e) \tag{6.148}$$

Rearranging equation (6.148) yields

$$\frac{\xi_{\mathcal{B}}\, P_{\mathcal{B}1}\, C_{v\mathcal{B}}}{P_{\mathcal{A}1}\, R_{\mathcal{B}}}\left(1 - \frac{T_{\mathcal{B}e}}{T_{\mathcal{B}1}}\right) = -f(\xi_e) \tag{6.149}$$

Notice that the temperature ratio cannot be substituted as it was done for the driver section because the process is not isentropic in the expansion section. However, the mass conservation can be applied as

$$\frac{T_{\mathcal{B}e}}{T_{\mathcal{B}1}} = \frac{P_{\mathcal{B}e}\, (\xi_{\mathcal{B}} - \xi_e)}{P_{\mathcal{B}1}\xi_{\mathcal{B}}} \tag{6.150}$$

Substituting equation (6.150) into equation (6.149) yields

$$\xi_{\mathcal{B}}\, \frac{P_{\mathcal{B}1}\, C_{v\mathcal{B}}}{P_{\mathcal{A}1} R_{\mathcal{B}}}\left(1 - \frac{P_{\mathcal{B}e}\, (\xi_{\mathcal{B}} - \xi_e)}{P_{\mathcal{B}1}\xi_{\mathcal{B}}}\right) = -f(\xi_e) \tag{6.151}$$

The equilibrium requirement dictates that the $P_{\mathcal{B}e} = P_{\mathcal{A}e}$. Thus, equation (6.151) and equation (6.145) have only two unknowns, ξ and $P_{\mathcal{A}e}$ and mathematically are solvable. The solution provides the displacement and the final pressure which are related

[17] This point refers to how much energy can be extracted from a given conditions.

[18] It must be noted that the expression for the work, $f(\xi_e)$ can be derived for the expansion chamber. The pressure in the expansion chamber is known and hence the integration can be carried for the process. It is assumed that the work is transferred to that chamber because energy is conserved.

to the final temperature. With the expression for the work and the displacement value, the maximum temperature can be calculated. The solution can be obtained by numerous methods to solve two nonlinear equations and no discussion is needed on this point.

In the solution, the displacement was found first and later the relationship between the displacement and pressure which was obtained by manipulating equation (6.145) to be

$$\frac{P_{\mathcal{A}e}}{P_{\mathcal{A}1}} = \left(1 - \frac{f(\xi_e\ (k_{\mathcal{A}} - 1)}{\xi_{\mathcal{A}}}\right) \stackrel{def}{\equiv} g(\xi_e) \tag{6.152}$$

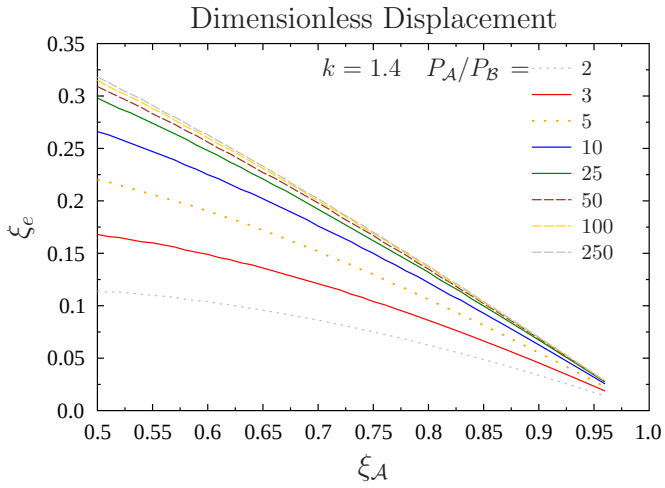


Fig. -6.30. The equilibrium length as a function of the initial dimensionless length.

The results are presented in Figures 6.30 and Figure 6.31. A shock tube operates close to the model will produce a larger temperature increase in the expansion shock. The ruptured diaphragm, the common method used in shock tube, on the other hand has inherently two main causes of the energy loss. The strongest deviation from the ideal isentropic model, suggested by this model, is the "penetration" of the reflecting shocks into the driver section as seen in Figure 6.32 by the light purple color in the middle. In this zone, the energy which was supposed to heat the driven

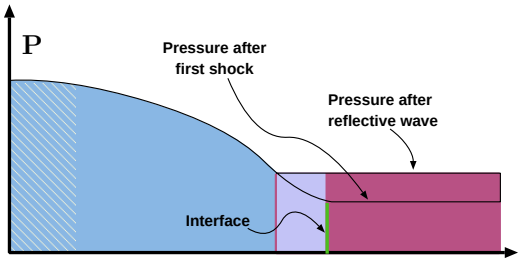


Fig. -6.32. Explanation why the ruptured diaphragm cannot reach maximum temperature.

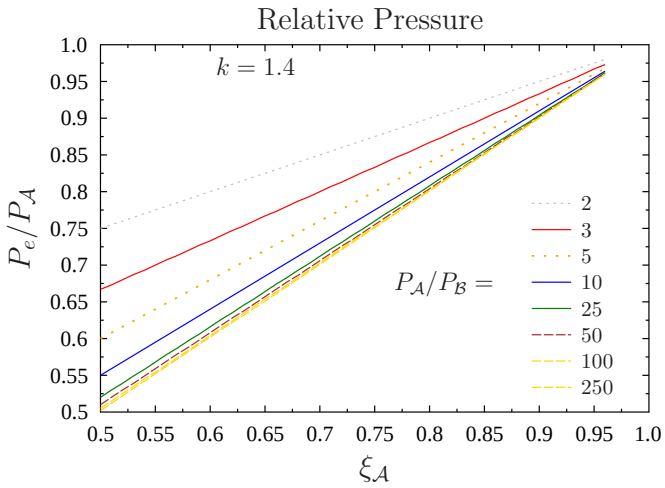


Fig. -6.31. The equilibrium pressure as a function of the initial dimensionless length.

section enters into the driver section. This fact reduces the potential energy in the driven section hence the maximum temperature. The second cause is the large part of the driver section such as the diagonal lines marked in Figure 6.32. Thus making infinitely long driver section does not produces any additional increase of the driven section temperature.

The secondary cause is the shocks and strong sound waves (minor shock waves) which appears in the entire zone of the driver section. The difference between the two causes is that the first one takes energy from the expansion section while the second one does not but increase the entropy in the driver section.

Example 6.15:
If the diaphragm in example 6.14 is the ruptured, estimate the maximum temperature. Specify the number of shocks you are using and explain why this number was selected. Assume that the pressure in the driven section is 1[Bar] and the temperature in the whole shock tube is 300° C.

SOLUTION

Under construction

End Solution

Example 6.16:
In example 6.14 the exact value of the dimensionless was not calculated but rather estimated. In this example calculate the value of the expansion. The shock tube radius is $0.1[m]$ *and the length ratio of the driver section to deriven section is 1.0. The initial*

pressure at the driver section is 1000 [Bar] and its length is 2 [m]. Calculate the energy available in the driver section. Calculate the dimensionless equilibrium expansion length ξ_e and the equilibrium pressure.

SOLUTION

Now the value of dimensionless expansion length, ξ_e can be found. Utilizing the value length ratio it can be obtained that

$$\xi_{\mathcal{A}} = \frac{1}{1 + \frac{1}{1.0}} = 0.5$$

The pressure ratio is 1000 and with $\xi_{\mathcal{A}}$ one can get that $\xi_e \sim 0.32$ from Figure 6.30 . Notice that the value for the pressure ratio 1000 is not given in the figure. However, the values for pressure ratio above 250 and above are converging to the same values. The expansion length is

$$L = L_{\mathcal{A}} \left(1 + \frac{L_{\mathcal{B}}}{L_{\mathcal{A}}}\right) = 2 \times 2 = 4[m]$$

With this value, the potential energy or work (see Figure 6.29 the red line) is

$$W = A\, L\, P_{\mathcal{A}1} f(\xi) = \pi \times 0.1^2 \times 4 \times 1000\, 10^5 \times 1000\, 10^5 \times 0.3 \sim 3769.9[kJ]$$

The equilibrium pressure is according to Figure 6.29 the pressure ration 0.67. Hence the pressure ratio is

$$P_e = P_{\mathcal{A}} \frac{P_e}{P_{\mathcal{A}}} = 1000[Bar] \times 0.67 = 670[Bar]$$

Notice that this pressure significantly lower than the pressure obtained after a shock or even a reflective shock.

End Solution

Example 6.17:
The conditions in the driven section attached to the driver of Example 6.16 are the pressure is 1[Bar] and specific heat is 0.7 $[kj/kg^{\circ}\mathrm{C}]$. Calculate the final temperature if there is not heat transfer. Assume that the universal constant $R = 287 j/kg^{\circ}\mathrm{C}$.

SOLUTION

The transfer energy was calculated in Example 6.16 as 3769.9 [kJ]. The mass in the driven chamber is

$$m_{\mathcal{B}} = \frac{P\,V}{R\,T} = \frac{100,000 \times \pi\, 0.1^2 \times \overbrace{2}^{L_{\mathcal{B}}}}{287 \times 300} = 0.073kg$$

The energy balance on the deriven section expressed by equation (6.148)

$$T_{Be} = T_{B2} + \frac{A\,L\,P_{A1}\,f(\xi_e)}{m\,C_p} = \frac{3769.9}{0.073 \times 0.7} \sim 73775[K]$$

This value is much higher that will be expect in traditional ruptured diaphragm.

End Solution

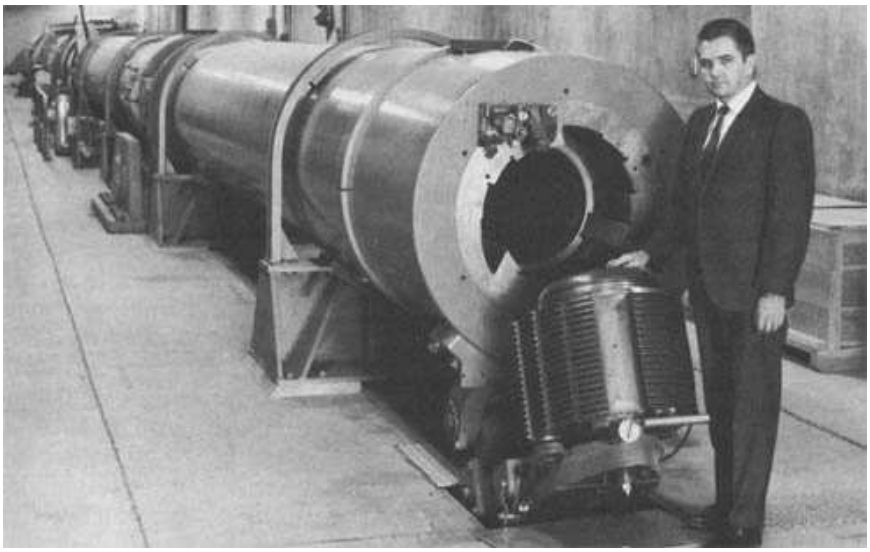


Fig. -6.33. Ames Research Center Shock Tube with Thomas N. Canning.

Example 6.18:
In Figure 6.33 showing a shock tube with driver with (32 ft) long and diameter (13in). The expansion diameter is 48 ft long and 7in diameter. The operational pressure in some cases was 250 [Bar] and the absolute pressure in the driven section was about 0.25 [Bar]. Calculate the energy lost using the ruptured diaphragm instead of using the moving diaphragm. You can assume that only the initial shock and the first reflecting shock contributing to the temperature increase and the rest is used solely to increase the entropy.

Supplemental Problems

1. In the analysis of the maximum temperature in the shock tube, it was assumed that process is isentropic. If this assumption is not correct would the maximum temperature obtained is increased or decreased?

2. In the analysis of the maximum temperature in the shock wave it was assumed

that process is isentropic. Clearly, this assumption is violated when there are shock waves. In that cases, what is the reasoning behind use this assumption any why?

6.5 Shock with Real Gases

6.6 Shock in Wet Steam

6.7 Normal Shock in Ducts

The flow in ducts is related to boundary layer issues. For a high Reynolds number, the assumption of an uniform flow in the duct is closer to reality. It is normal to have a large Mach number with a large Re number. In that case, the assumptions in construction of these models are acceptable and reasonable.

6.8 Additional Examples for Moving Shocks

Example 6.19:

This problem was taken from the real industrial manufacturing world. An engineer is required to design a cooling system for a critical electronic device. The temperature should not increase above a certain value. In this system, air is supposed to reach the pipe exit as quickly as possible when the valve is opened (see Figure (6.34)*). The distance between between the valve and the pipe exit is* $3[m]$*. The conditions upstream of the valve are* $30[Bar]$ *and* $27^{\circ}\mathrm{C}$ *. Assume that there isn't any resistance whatsoever in the pipe. The ambient temperature is* $27^{\circ}\mathrm{C}$ *and* $1[Bar]$*. Assume that the time scale for opening the valve is significantly smaller than the typical time of the pipe (totally unrealistic even though the valve manufacture claims of 0.0002 [sec] valve to be completely opened). After building the system, the engineer notices that the system does not cool the device fast enough and proposes to increase the pressure and increase the diameter of the pipe. Comment on this proposal. Where any of these advises make any sense in the light of the above assumptions? What will be your recommendations to the manufacturing company? Plot the exit temperature and the mass flow rate as a function of the time.*

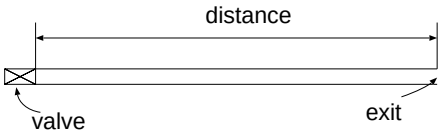


Fig. -6.34. Figure for Example (6.19)

SOLUTION

This problem is known as the suddenly open valve problem in which the shock choking phenomenon occurs. The time it takes for the shock to travel from the valve depends on the pressure ratio $\frac{P_y}{P_x} = 30$ for which the following table is obtained

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
5.0850	0.41404	0.0	1.668	5.967	30.00	0.057811

The direct calculation will be by using the "upstream" Mach number, $M_x = M_{sx} = 5.0850$. Therefore, the time is

$$t = \frac{distance}{M_{sx}\sqrt{k\,R\,T_x}} = \frac{3}{5.0850\sqrt{1.4 \times 287 \times 300}} \sim 0.0017[sec]$$

The mass flow rate after reaching the exit under these assumptions remains constant until the uncooled material reaches the exit. The time it takes for the material from the valve to reach the exit is

$$t = \frac{distance}{M_y{}'\sqrt{k\,R\,T_y}} = \frac{3}{1.668\sqrt{1.4 \times 287 \times 300 \times 5.967}} \sim 0.0021[sec]$$

During that difference of time the material is get heated instead of cooling down because of the high temperature. The suggestion of the engineer to increase the pressure will decrease the time but will increase the temperature at the exit during this critical time period. Thus, this suggestion contradicts the purpose of the required manufacturing needs.

To increase the pipe diameter will not change the temperature and therefore will not change the effects of heating. It can only increase the rate after the initial heating spike

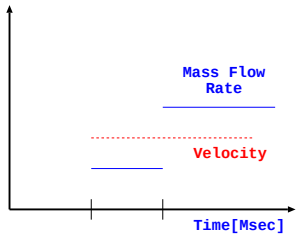


Fig. -6.35. The results for Example (6.19)

A possible solution is to have the valve very close to the pipe exit. Thus, the heating time is reduced significantly. There is also the possibility of steps increase in which every step heat released will not be enough to over heat the device. The last possible requirement a programmable valve and very fast which its valve probably exceed the moving shock the valve downstream. The plot of the mass flow rate and the velocity are given in Figure 6.35.

End Solution

Example 6.20:

Example (6.19) *deals with a damaging of electronic product by the temperature increase. Try to estimate the temperature increase of the product. Plot the pipe exit temperature as a function of the time.*

SOLUTION

To be developed

End Solution

6.9 Tables of Normal Shocks, $k = 1.4$ Ideal Gas

Table -6.2. The shock wave table for k = 1.4

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_y}{\rho_x}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.00	1.00000	1.00000	1.00000	1.00000	1.00000
1.05	0.95313	1.03284	1.08398	1.11958	0.99985
1.10	0.91177	1.06494	1.16908	1.24500	0.99893
1.15	0.87502	1.09658	1.25504	1.37625	0.99669
1.20	0.84217	1.12799	1.34161	1.51333	0.99280
1.25	0.81264	1.15938	1.42857	1.65625	0.98706
1.30	0.78596	1.19087	1.51570	1.80500	0.97937
1.35	0.76175	1.22261	1.60278	1.95958	0.96974
1.40	0.73971	1.25469	1.68966	2.12000	0.95819
1.45	0.71956	1.28720	1.77614	2.28625	0.94484
1.50	0.70109	1.32022	1.86207	2.45833	0.92979
1.55	0.68410	1.35379	1.94732	2.63625	0.91319
1.60	0.66844	1.38797	2.03175	2.82000	0.89520
1.65	0.65396	1.42280	2.11525	3.00958	0.87599
1.70	0.64054	1.45833	2.19772	3.20500	0.85572
1.75	0.62809	1.49458	2.27907	3.40625	0.83457
1.80	0.61650	1.53158	2.35922	3.61333	0.81268
1.85	0.60570	1.56935	2.43811	3.82625	0.79023
1.90	0.59562	1.60792	2.51568	4.04500	0.76736
1.95	0.58618	1.64729	2.59188	4.26958	0.74420
2.00	0.57735	1.68750	2.66667	4.50000	0.72087
2.05	0.56906	1.72855	2.74002	4.73625	0.69751
2.10	0.56128	1.77045	2.81190	4.97833	0.67420
2.15	0.55395	1.81322	2.88231	5.22625	0.65105

Table -6.2. The shock wave table for k = 1.4 (continue)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.20	0.54706	1.85686	2.95122	5.48000	0.62814
2.25	0.54055	1.90138	3.01863	5.73958	0.60553
2.30	0.53441	1.94680	3.08455	6.00500	0.58329
2.35	0.52861	1.99311	3.14897	6.27625	0.56148
2.40	0.52312	2.04033	3.21190	6.55333	0.54014
2.45	0.51792	2.08846	3.27335	6.83625	0.51931
2.50	0.51299	2.13750	3.33333	7.12500	0.49901
2.75	0.49181	2.39657	3.61194	8.65625	0.40623
3.00	0.47519	2.67901	3.85714	10.33333	0.32834
3.25	0.46192	2.98511	4.07229	12.15625	0.26451
3.50	0.45115	3.31505	4.26087	14.12500	0.21295
3.75	0.44231	3.66894	4.42623	16.23958	0.17166
4.00	0.43496	4.04688	4.57143	18.50000	0.13876
4.25	0.42878	4.44891	4.69919	20.90625	0.11256
4.50	0.42355	4.87509	4.81188	23.45833	0.09170
4.75	0.41908	5.32544	4.91156	26.15625	0.07505
5.00	0.41523	5.80000	5.00000	29.00000	0.06172
5.25	0.41189	6.29878	5.07869	31.98958	0.05100
5.50	0.40897	6.82180	5.14894	35.12500	0.04236
5.75	0.40642	7.36906	5.21182	38.40625	0.03536
6.00	0.40416	7.94059	5.26829	41.83333	0.02965
6.25	0.40216	8.53637	5.31915	45.40625	0.02498
6.50	0.40038	9.15643	5.36508	49.12500	0.02115
6.75	0.39879	9.80077	5.40667	52.98958	0.01798
7.00	0.39736	10.46939	5.44444	57.00000	0.01535

Table -6.2. The shock wave table for k = 1.4 (continue)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
7.25	0.39607	11.16229	5.47883	61.15625	0.01316
7.50	0.39491	11.87948	5.51020	65.45833	0.01133
7.75	0.39385	12.62095	5.53890	69.90625	0.00979
8.00	0.39289	13.38672	5.56522	74.50000	0.00849
8.25	0.39201	14.17678	5.58939	79.23958	0.00739
8.50	0.39121	14.99113	5.61165	84.12500	0.00645
8.75	0.39048	15.82978	5.63218	89.15625	0.00565
9.00	0.38980	16.69273	5.65116	94.33333	0.00496
9.25	0.38918	17.57997	5.66874	99.65625	0.00437
9.50	0.38860	18.49152	5.68504	105.12500	0.00387
9.75	0.38807	19.42736	5.70019	110.73958	0.00343
10.00	0.38758	20.38750	5.71429	116.50000	0.00304

Table -6.3. Table for a Reflective Shock from a suddenly closed end (k=1.4)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.006	0.99403	0.01	0.0	1.004	1.014	1.00000
1.012	0.98812	0.02	0.0	1.008	1.028	1.00000
1.018	0.98227	0.03	0.0	1.012	1.043	0.99999
1.024	0.97647	0.04	0.0	1.016	1.057	0.99998
1.030	0.97074	0.05	0.0	1.020	1.072	0.99997
1.037	0.96506	0.06	0.0	1.024	1.087	0.99994
1.043	0.95944	0.07	0.0	1.028	1.102	0.99991
1.049	0.95387	0.08	0.0	1.032	1.118	0.99986
1.055	0.94836	0.09	0.0	1.036	1.133	0.99980
1.062	0.94291	0.10	0.0	1.040	1.149	0.99973

Table -6.3. Table for Reflective Shock from suddenly closed valve (end) (k=1.4)(continue)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{Po_y}}{\mathbf{Po_x}}$
1.127	0.89128	0.20	0.0	1.082	1.316	0.99790
1.196	0.84463	0.30	0.0	1.126	1.502	0.99317
1.268	0.80251	0.40	0.0	1.171	1.710	0.98446
1.344	0.76452	0.50	0.0	1.219	1.941	0.97099
1.423	0.73029	0.60	0.0	1.269	2.195	0.95231
1.505	0.69946	0.70	0.0	1.323	2.475	0.92832
1.589	0.67171	0.80	0.0	1.381	2.780	0.89918
1.676	0.64673	0.90	0.0	1.442	3.112	0.86537
1.766	0.62425	1.00	0.0	1.506	3.473	0.82755
1.858	0.60401	1.10	0.0	1.576	3.862	0.78652
1.952	0.58578	1.20	0.0	1.649	4.280	0.74316
2.048	0.56935	1.30	0.0	1.727	4.728	0.69834
2.146	0.55453	1.40	0.0	1.810	5.206	0.65290
2.245	0.54114	1.50	0.0	1.897	5.715	0.60761
2.346	0.52904	1.60	0.0	1.990	6.256	0.56312
2.448	0.51808	1.70	0.0	2.087	6.827	0.51996
2.552	0.50814	1.80	0.0	2.189	7.431	0.47855
2.656	0.49912	1.90	0.0	2.297	8.066	0.43921
2.762	0.49092	2.00	0.0	2.410	8.734	0.40213
3.859	0.43894	3.00	0.0	3.831	17.21	0.15637
5.000	0.41523	4.00	0.0	5.800	29.00	0.061716
6.162	0.40284	5.00	0.0	8.325	44.14	0.026517
7.336	0.39566	6.00	0.0	11.41	62.62	0.012492
8.517	0.39116	7.00	0.0	15.05	84.47	0.00639
9.703	0.38817	8.00	0.0	19.25	$1.1E+2$	0.00350

Table -6.3. Table for Reflective Shock from suddenly closed valve (end) (k=1.4)(continue)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
10.89	0.38608	9.00	0.0	24.01	$1.4E{+}2$	0.00204
12.08	0.38457	10.0	0.0	29.33	$1.7E{+}2$	0.00125

Table -6.4. Table for shock propagating from suddenly **opened** *valve (k=1.4)*

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.006	0.99402	0.0	0.01	1.004	1.014	1.00000
1.012	0.98807	0.0	0.02	1.008	1.028	1.00000
1.018	0.98216	0.0	0.03	1.012	1.043	0.99999
1.024	0.97629	0.0	0.04	1.016	1.058	0.99998
1.031	0.97045	0.0	0.05	1.020	1.073	0.99996
1.037	0.96465	0.0	0.06	1.024	1.088	0.99994
1.044	0.95888	0.0	0.07	1.029	1.104	0.99990
1.050	0.95315	0.0	0.08	1.033	1.120	0.99985
1.057	0.94746	0.0	0.09	1.037	1.136	0.99979
1.063	0.94180	0.0	0.10	1.041	1.152	0.99971
1.133	0.88717	0.0	0.20	1.086	1.331	0.99763
1.210	0.83607	0.0	0.30	1.134	1.541	0.99181
1.295	0.78840	0.0	0.40	1.188	1.791	0.98019
1.390	0.74403	0.0	0.50	1.248	2.087	0.96069
1.495	0.70283	0.0	0.60	1.317	2.441	0.93133
1.613	0.66462	0.0	0.70	1.397	2.868	0.89039
1.745	0.62923	0.0	0.80	1.491	3.387	0.83661
1.896	0.59649	0.0	0.90	1.604	4.025	0.76940
2.068	0.56619	0.0	1.00	1.744	4.823	0.68907
2.269	0.53817	0.0	1.100	1.919	5.840	0.59699

Table -6.4. Table for shock propagating from suddenly **opened** *valve (k=1.4)*

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.508	0.51223	0.0	1.200	2.145	7.171	0.49586
2.799	0.48823	0.0	1.300	2.450	8.975	0.38974
3.167	0.46599	0.0	1.400	2.881	11.54	0.28412
3.658	0.44536	0.0	1.500	3.536	15.45	0.18575
4.368	0.42622	0.0	1.600	4.646	22.09	0.10216
5.551	0.40843	0.0	1.700	6.931	35.78	0.040812
8.293	0.39187	0.0	1.800	14.32	80.07	0.00721
8.821	0.39028	0.0	1.810	16.07	90.61	0.00544
9.457	0.38870	0.0	1.820	18.33	$1.0E+2$	0.00395
10.24	0.38713	0.0	1.830	21.35	$1.2E+2$	0.00272
11.25	0.38557	0.0	1.840	25.57	$1.5E+2$	0.00175
12.62	0.38402	0.0	1.850	31.92	$1.9E+2$	0.00101
14.62	0.38248	0.0	1.860	42.53	$2.5E+2$	0.000497
17.99	0.38096	0.0	1.870	63.84	$3.8E+2$	0.000181
25.62	0.37944	0.0	1.880	$1.3E+2$	$7.7E+2$	$3.18E-5$
61.31	0.37822	0.0	1.888	$7.3E+2$	$4.4E+3$	0.0
62.95	0.37821	0.0	1.888	$7.7E+2$	$4.6E+3$	0.0
64.74	0.37820	0.0	1.888	$8.2E+2$	$4.9E+3$	0.0
66.69	0.37818	0.0	1.888	$8.7E+2$	$5.2E+3$	0.0
68.83	0.37817	0.0	1.888	$9.2E+2$	$5.5E+3$	0.0
71.18	0.37816	0.0	1.889	$9.9E+2$	$5.9E+3$	0.0
73.80	0.37814	0.0	1.889	$1.1E+3$	$6.4E+3$	0.0
76.72	0.37813	0.0	1.889	$1.1E+3$	$6.9E+3$	0.0
80.02	0.37812	0.0	1.889	$1.2E+3$	$7.5E+3$	0.0
83.79	0.37810	0.0	1.889	$1.4E+3$	$8.2E+3$	0.0

Table -6.5. Table for shock propagating from a suddenly **opened** *valve (k=1.3)*

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
1.0058	0.99427	0.0	0.010	1.003	1.013	1.00000
1.012	0.98857	0.0	0.020	1.006	1.026	1.00000
1.017	0.98290	0.0	0.030	1.009	1.040	0.99999
1.023	0.97726	0.0	0.040	1.012	1.054	0.99998
1.029	0.97166	0.0	0.050	1.015	1.067	0.99997
1.035	0.96610	0.0	0.060	1.018	1.081	0.99995
1.042	0.96056	0.0	0.070	1.021	1.096	0.99991
1.048	0.95506	0.0	0.080	1.024	1.110	0.99987
1.054	0.94959	0.0	0.090	1.028	1.125	0.99981
1.060	0.94415	0.0	0.100	1.031	1.140	0.99975
1.126	0.89159	0.0	0.200	1.063	1.302	0.99792
1.197	0.84227	0.0	0.300	1.098	1.489	0.99288
1.275	0.79611	0.0	0.400	1.136	1.706	0.98290
1.359	0.75301	0.0	0.500	1.177	1.959	0.96631
1.452	0.71284	0.0	0.600	1.223	2.252	0.94156
1.553	0.67546	0.0	0.700	1.274	2.595	0.90734
1.663	0.64073	0.0	0.800	1.333	2.997	0.86274
1.785	0.60847	0.0	0.900	1.400	3.471	0.80734
1.919	0.57853	0.0	1.00	1.478	4.034	0.74136
2.069	0.55074	0.0	1.100	1.570	4.707	0.66575
2.236	0.52495	0.0	1.200	1.681	5.522	0.58223
2.426	0.50100	0.0	1.300	1.815	6.523	0.49333
2.644	0.47875	0.0	1.400	1.980	7.772	0.40226
2.898	0.45807	0.0	1.500	2.191	9.367	0.31281
3.202	0.43882	0.0	1.600	2.467	11.46	0.22904

Table -6.5. Table for shock propagating from a suddenly **opened** *valve (k=1.3)*

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.576	0.42089	0.0	1.700	2.842	14.32	0.15495
4.053	0.40418	0.0	1.800	3.381	18.44	0.093988
4.109	0.40257	0.0	1.810	3.448	18.95	0.088718
4.166	0.40097	0.0	1.820	3.519	19.49	0.083607
4.225	0.39938	0.0	1.830	3.592	20.05	0.078654
4.286	0.39780	0.0	1.840	3.669	20.64	0.073863
4.349	0.39624	0.0	1.850	3.749	21.25	0.069233
4.415	0.39468	0.0	1.860	3.834	21.90	0.064766
4.482	0.39314	0.0	1.870	3.923	22.58	0.060462
4.553	0.39160	0.0	1.880	4.016	23.30	0.056322
4.611	0.39037	0.0	1.888	4.096	23.91	0.053088
4.612	0.39035	0.0	1.888	4.097	23.91	0.053053
4.613	0.39034	0.0	1.888	4.098	23.92	0.053018
4.613	0.39033	0.0	1.888	4.099	23.93	0.052984
4.614	0.39031	0.0	1.888	4.099	23.93	0.052949
4.615	0.39030	0.0	1.889	4.100	23.94	0.052914
4.615	0.39029	0.0	1.889	4.101	23.95	0.052879
4.616	0.39027	0.0	1.889	4.102	23.95	0.052844
4.616	0.39026	0.0	1.889	4.103	23.96	0.052809
4.617	0.39025	0.0	1.889	4.104	23.97	0.052775

CHAPTER 7

Normal Shock in Variable Duct Areas

In the previous two chapters, the flow in a variable area duct and the normal shock (discontinuity) were discussed. A discussion of the occurrences of shock in flow in a variable is presented. As it is was presented before, the shock can occur **only** in steady state when there is a supersonic flow. but also in steady state cases when there is no supersonic flow (in stationary coordinates). As it was shown in Chapter 6, the gas has to pass through a converging–diverging nozzle to obtain a supersonic flow.

In the previous chapter, the flow in a convergent–divergent nuzzle was presented when the pressure ratio was above or below the special range. This Chapter will present the flow in this special range of pressure ratios. It is interesting to note that a normal shock must occur in these situations (pressure ratios).

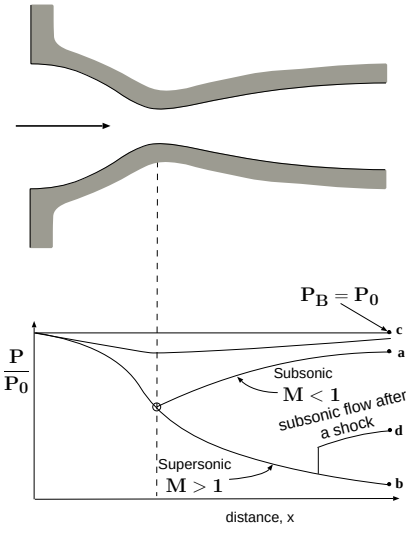


Fig. -7.1. The flow in the nozzle with different back pressures.

In Figure 7.1 the reduced pressure distribution in the converging–diverging nozzle is shown in its whole range of pressure ratios. When the pressure ratio, P_B is between point "**a**" and point "**b**" the flow is different from what was discussed before. In this

case, no continuous pressure possibly can exists. Only in one point where $P_B = P_b$ continuous pressure exist. If the back pressure, P_B is smaller than P_b a discontinuous point (a shock) will occur. In conclusion, once the flow becomes supersonic, only exact geometry can achieve continuous pressure flow.

In the literature, some refer to a nozzle with an area ratio such point **b** as above the back pressure and it is referred to as an under–expanded nozzle. In the under–expanded case, the nozzle doesn't provide the maximum thrust possible. On the other hand, when the nozzle exit area is too large a shock will occur and other phenomenon such as plume will separate from the wall inside the nozzle. This nozzle is called an over–expanded nozzle. In comparison of nozzle performance for rocket and aviation, the over–expanded nozzle is worse than the under–expanded nozzle because the nozzle's large exit area results in extra drag.

The location of the shock is determined by geometry to achieve the right back pressure. Obviously if the back pressure, P_B, is lower than the critical value (the only value that can achieve continuous pressure) a shock occurs outside of the nozzle. If the back pressure is within the range of P_a to P_b than the exact location determines that after the shock the subsonic branch will match the back pressure.

The first example is for academic reasons. It has to be recognized that the shock wave isn't easily visible (see Mach's photography techniques). Therefore, this example provides a demonstration of the calculations required for the location even if it isn't realistic. Nevertheless, this example will provide the fundamentals to explain the usage of the tools (equations and tables) that were developed so far.

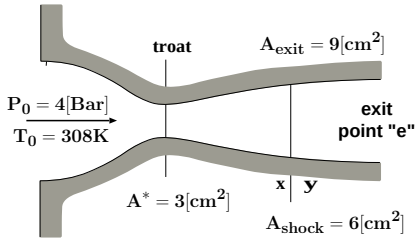


Fig. -7.2. A nozzle with normal shock.

Example 7.1:

*A large tank with compressed air is attached into a converging–diverging nozzle at pressure $4[Bar]$ and temperature of35°C. Nozzle throat area is 3[cm^2] and the exit area is 9[cm^2]. The shock occurs in a location where the cross section area is 6[cm^2]. Calculate the back pressure and the temperature of the flow. (It should be noted that the temperature of the surrounding is irrelevant in this case.) Also determine the critical points for the back pressure (point "***a***" and point "***b***").*

SOLUTION

Since the key word "large tank" was used that means that the stagnation temperature and pressure are known and equal to the conditions in the tank.

First, the exit Mach number has to be determined. This Mach number can be calculated by utilizing the isentropic relationship from the large tank to the shock (point **"x"**). Then the relationship developed for the shock can be utilized to calculate the Mach number after the shock, (point **"y"**). From the Mach number after the shock, M_y, the Mach number at the exit can be calculated by utilizing the isentropic

relationship.

It has to be realized that for a large tank, the inside conditions are essentially the stagnation conditions (this statement is said without a proof, but can be shown that the correction is negligible for a typical dimension ratio that is over 100. For example, in the case of ratio of 100 the Mach number is 0.00587 and the error is less than %0.1). Thus, the stagnation temperature and pressure are known $T_0 = 308K$ and $P_0 = 4[Bar]$. The star area (the throat area), A^*, before the shock is known and given as well.

$$\frac{A_x}{A^*} = \frac{6}{3} = 2$$

With this ratio $(A/A^* = 2)$ utilizing the Table (6.2) or equation (5.47) or the GDC–Potto, the Mach number, M_x is about 2.197 as shown table below:

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
2.1972	0.50877	0.18463	2.0000	0.09393	0.18787

With this Mach number, $M_x = 2.1972$ the Mach number, M_y can be obtained. From equation (6.22) or from Table (5.2) $M_y \cong 0.54746$. With these values, the subsonic branch can be evaluated for the pressure and temperature ratios.

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.1972	0.54743	1.8544	2.9474	5.4656	0.62941

From Table (5.2) or from equation (5.11) the following Table for the isentropic relationship is obtained

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
0.54743	0.94345	0.86457	1.2588	0.81568	1.0268

Again utilizing the isentropic relationship the exit conditions can be evaluated. With known Mach number the new star area ratio, A_y/A^* is known and the exit area can be calculated as

$$\frac{A_e}{A^*} = \frac{A_e}{A_y} \times \frac{A_y}{A^*} = 1.2588 \times \frac{9}{6} = 1.8882$$

with this area ratio, $\frac{A_e}{A^*} = 1.8882$, one can obtain using the isentropic relationship as

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
0.32651	0.97912	0.94862	1.8882	0.92882	1.7538

Since the stagnation pressure is constant as well the stagnation temperature, the exit conditions can be calculated.

$$\begin{aligned} P_{exit} =& \left(\frac{P_{exit}}{P_0}\right)\left(\frac{P_0}{P_y}\right)\left(\frac{P_y}{P_x}\right)\left(\frac{P_x}{P_0}\right)P_0 \\ =& 0.92882 \times \left(\frac{1}{0.81568}\right) \times 5.466 \times 0.094 \times 4 \\ \cong& 2.34[Bar] \end{aligned}$$

The exit temperature is

$$\begin{aligned} T_{exit} =& \left(\frac{T_{exit}}{T_0}\right)\left(\frac{T_0}{T_y}\right)\left(\frac{T_y}{T_x}\right)\left(\frac{T_x}{T_0}\right)T_0 \\ =& 0.98133 \times \left(\frac{1}{0.951}\right) \times 1.854 \times 0.509 \times 308 \\ \cong& 299.9K \end{aligned}$$

For the "critical" points "a" and "b" are the points that the shock doesn't occur and yet the flow achieve Mach equal 1 at the throat. In that case we don't have to go through that shock transition. Yet we have to pay attention that there two possible back pressures that can "achieve" it or target. The area ratio for both cases, is $A/A^* = 3$ In the subsonic branch (either using equation or the isentropic table or GDC-Potto as

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
0.19745	0.99226	0.98077	3.0000	0.97318	2.9195
2.6374	0.41820	0.11310	3.0000	0.04730	0.14190

$$P_{exit} = \left(\frac{P_{exit}}{P_0}\right)P_0 = 0.99226 \times 4 \cong 3.97[Bar]$$

For the supersonic sonic branch

$$P_{exit} = \left(\frac{P_{exit}}{P_0}\right)P_0 = 0.41820 \times 4 \cong 1.6728[Bar]$$

It should be noted that the flow rate is constant and maximum for any point beyond

the point "a" even if the shock is exist. The flow rate is expressed as following

$$\dot{m} = \rho^* A^* U = \overbrace{\frac{P^*}{RT^*}}^{\rho^*} A \overbrace{cM}^{M=1} = \frac{\left(\overbrace{\frac{P^*}{P_0}P_0}^{P^*}\right)}{R\left(\underbrace{\frac{T^*}{T_0}T_0}_{T^*}\right)} A \overbrace{\sqrt{kit^*}}^{c} = \frac{\left(\frac{P^*}{P_0}P_0\right)}{R\left(\frac{T^*}{T_0}T_0\right)} A \sqrt{k\,R\,\frac{T^*}{T_0}T_0}$$

The temperature at the throat is:

$$T^* = \left(\frac{T^*}{T_0}\right) T_0 = 0.833 \times 308 = 256.7K$$

The temperature at the throat reads

$$P^* = \frac{(P^* P_0)\, P_0}{=} 0.5283 \times 4 = 2.113[Bar]$$

The speed of sound is

$$c = \sqrt{1.4 \times 287 \times 256.7} = 321.12[m/sec]$$

And the mass flow rate reads

$$\dot{m} = \frac{410^5}{287 \times 256.7} 3 \times 10^{-4} \times 321.12 = 0.13[kg/sec]$$

It is interesting to note that in this case the choking condition is obtained ($M = 1$) when the back pressure just reduced to less than 5% than original pressure (the pressure in the tank). While the pressure to achieve full supersonic flow through the nozzle the pressure has to be below the 42% the original value. Thus, over 50% of the range of pressure a shock occurs some where in the nozzle. In fact in many industrial applications, these kind situations exist. In these applications a small pressure difference can produce a shock wave and/or a chock flow.

End Solution

For more practical example[1] from industrial application point of view.

Example 7.2:
In the data from the above example (7.1) *where would be shock's location when the back pressure is* $2[Bar]$*?*

[1] The meaning of the word practical is that in reality the engineer does not given the opportunity to determine the location of the shock but rather information such as pressures and temperature.

SOLUTION

The solution procedure is similar to what was shown in previous Example (7.1). The solution process starts at the nozzle's exit and progress to the entrance.

The conditions in the tank are again the stagnation conditions. Thus, the exit pressure is between point "a" and point "b". It follows that there must exist a shock in the nozzle. Mathematically, there are two main possible ways to obtain the solution. In the first method, the previous example information used and expanded. In fact, it requires some iterations by "smart" guessing the different shock locations. The area (location) that the previous example did not "produce" the "right" solution (the exit pressure was $2.113[Bar]$. Here, the needed pressure is only $2[Bar]$ which means that the next guess for the shock location should be with a larger area[2]. The second (recommended) method is noticing that the flow is adiabatic and the mass flow rate is constant which means that the ratio of the $P_0 \times A^* = P_{y0} \times A^*|_{@y}$ (upstream conditions are known, see also equation (5.70)).

$$\frac{P_{exit}A_{exit}}{P_{x0} \times {A_x}^*} = \frac{P_{exit}A_{exit}}{P_{y_0} \times {A_y}^*} = \frac{2 \times 9}{4 \times 3} = 1.5[unitless!]$$

With the knowledge of the ratio $\frac{PA}{P_0 A^*}$ which was calculated and determines the exit Mach number. Utilizing the Table (5.2) or the GDC-Potto provides the following table is obtained

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.38034	0.97188	0.93118	1.6575	0.90500	1.5000	0.75158

With these values the relationship between the stagnation pressures of the shock are obtainable e.g. the exit Mach number, M_y, is known. The exit total pressure can be obtained (if needed). More importantly the pressure ratio exit is known. The ratio of the ratio of stagnation pressure obtained by

$$\frac{P_{0y}}{P_{0x}} = \overbrace{\left(\frac{P_{0y}}{P_{exit}}\right)}^{for\ M_{exit}} \left(\frac{P_{exit}}{P_{0x}}\right) = \frac{1}{0.905} \times \frac{2}{4} = 0.5525$$

Looking up in the Table (5.2) or utilizing the GDC-Potto provides

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.3709	0.52628	2.0128	3.1755	6.3914	0.55250

With the information of Mach number (either M_x or M_y) the area where the shock (location) occurs can be found. First, utilizing the isentropic Table (5.2).

[2] Of course, the computer can be use to carry this calculations in a sophisticate way.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
2.3709	0.47076	0.15205	2.3396	0.07158	0.16747

Approaching the shock location from the upstream (entrance) yields

$$A = \frac{A}{A*}A^* = 2.3396 \times 3 \cong 7.0188[cm^2]$$

Note, as "simple" check this value is larger than the value in the previous example.

End Solution

7.1 Nozzle efficiency

Obviously nozzles are not perfectly efficient and there are several ways to define the nozzle efficiency. One of the effective way is to define the efficiency as the ratio of the energy converted to kinetic energy and the total potential energy could be converted to kinetic energy. The total energy that can be converted is during isentropic process is

$$E = h_0 - h_{exits} \tag{7.1}$$

where h_{exits} is the enthalpy if the flow was isentropic. The actual energy that was used is

$$E = h_0 - h_{exit} \tag{7.2}$$

The efficiency can be defined as

$$\eta = \frac{h_0 - h_{exit}}{h_0 - h_{exits}} = \frac{(U_{actual})^2}{(U_{ideal})^2} \tag{7.3}$$

The typical efficiency of nozzle is ranged between 0.9 to 0.99. In the literature some define also velocity coefficient as the ratio of the actual velocity to the ideal velocity, V_c

$$V_c = \sqrt{\eta} = \sqrt{\frac{(U_{actual})^2}{(U_{ideal})^2}} \tag{7.4}$$

There is another less used definition which referred as the coefficient of discharge as the ratio of the actual mass rate to the ideal mass flow rate.

$$C_d = \frac{\dot{m}_{actual}}{\dot{m}_{ideal}} \tag{7.5}$$

Example 7.3:
A nozzle is known to operate at $\eta = 90\%$ efficiency when the flow is shock–less. The stagnation temperature entering to nozzle is 300 [K] with pressure 5[Bar]. The exit static pressure is 1[Bar]. The specific heat, k, is 1.4. Calculate the exit temperature, pressure, and Mach number of when the flow is shock free and isentropic. If the exit pressure is higher by 10%, what is the nozzle efficiency? Calculate the exit temperature, and Mach number for actual case. Is this results reasonable? Calculate the location of the shock, assuming the flow isentropic in the before the shock. Assume that the efficiency is 90% all location except the close vicinity of the shock. What the efficiency of the nozzle with the shock? Calculate the the velocity coefficient and the coefficient of discharge.

SOLUTION

The isentropic flow must be supersonic because the pressure ratio is below the critical point of 0.528282. Hence the conditions at the exit are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
1.7085	0.63139	0.31676	1.3457	0.20000	0.26914	0.57044

The velocity at the exit has to be calculated.

$$\begin{aligned} U &= M\sqrt{k\,R\,T} = M\sqrt{k\,R\,T_0\,\frac{T}{T_0}} \\ &= 1.7085\sqrt{1.4\times 287\times 300\times 0.63139} \\ &= 275.87[m/sec] \end{aligned}$$

If the velocity due to various reasons is not isentropic than the efficiency must be used.

$$U_{actual} = \sqrt{\eta}\,U_{ideal} = \sqrt{0.9}\times 275.87 \sim 261.72$$

The actual exit temperature is obtained from

$$T_exit = T_0 - \eta\,(T_0 - T_{exits}) = 300 - 0.9\times(300 - 300\times 0.63139) = 200.47K$$

The actual exit Mach number is

$$M_{exit} = \frac{U}{\sqrt{k\,R\,T}} = \frac{261.72}{\sqrt{1.4\times 287\times 200.47}} = 0.922$$

Which means that there are different velocities in the throat and other places as well.

End Solution

7.2 Diffuser Efficiency

The efficiency of the diffuser is defined as the ratio of the enthalpy change that occurred between the entrance to exit stagnation pressure to the kinetic energy.

$$\eta = \frac{2(h_3 - h_1)}{{U_1}^2} = \frac{h_3 - h_1}{h_{01} - h_1} \qquad (7.6)$$

For perfect gas equation (7.6) can be converted to

$$\eta = \frac{2\,C_p\,(T_3 - T_1)}{{U_1}^2} \qquad (7.7)$$

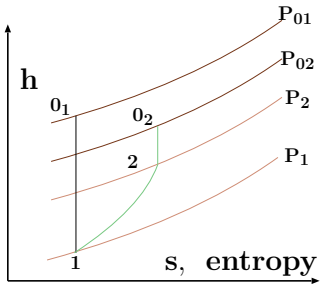


Fig. -7.3. Description to clarify the definition of diffuser efficiency.

And further expanding equation (7.7) results in

$$\eta = \frac{2\frac{kR}{k-1}T_1\left(\frac{T_3}{T_1} - 1\right)}{{c_1}^2{M_1}^2} = \frac{\frac{2}{k-1}\left(\frac{T_3}{T_1} - 1\right)}{{M_1}^2} = \frac{2}{{M_1}^2(k-1)}\left(\left(\frac{T_3}{T_1}\right)^{\frac{k-1}{k}} - 1\right) \qquad (7.8)$$

Example 7.4:

A wind tunnel combined from a nozzle and a diffuser (actually two nozzles connected by a constant area see Figure (7.4)) the required condition at point 3 are: $M = 3.0$ and pressure of $0.7[Bar]$ and temperature of $250K$. The cross section in area between the nuzzle and diffuser is $0.02[m^2]$. What is area of nozzle's throat and what is area of the diffuser's throat to maintain chocked diffuser with subsonic flow in the expansion section. $k = 1.4$ can be assumed. Assume that a shock occurs in the test section.

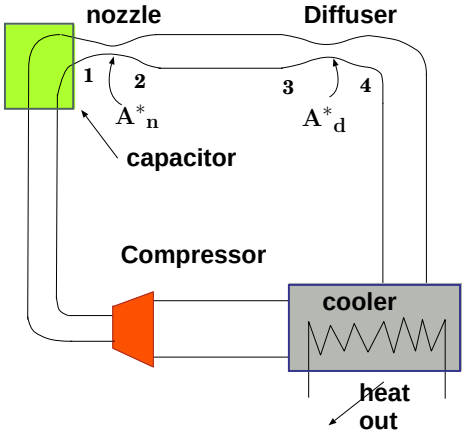


Fig. -7.4. Schematic of a supersonic tunnel in a continuous region (and also for Example (7.4).

SOLUTION

The condition at $M = 3$ is summarized in following table

M	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^*}$	$\frac{P}{P_0}$	$\frac{A\times P}{A^*\times P_0}$	$\frac{F}{F^*}$
3.0000	0.35714	0.07623	4.2346	0.02722	0.11528	0.65326

The nozzle area can be calculated by

$$A^*{}_n = \frac{A^\star}{A} A = 0.02/4.2346 = 0.0047[m^2]$$

In this case, $P_0 A^*$ is constant (constant mass flow). First the stagnation behind the shock will be

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.0000	0.47519	2.6790	3.8571	10.3333	0.32834

$$A^*{}_d = \frac{P_{0n}}{P_{0d}} A^*{}_n \sim \frac{1}{0.32834} 0.0047 \sim 0.0143[m^3]$$

End Solution

Example 7.5:
A shock is moving at 200 [m/sec] in pipe with gas with $k = 1.3$, pressure of 2 [Bar] and temperature of $350K$. Calculate the conditions after the shock.

SOLUTION

This is a case of completely and suddenly open valve with the shock velocity, temperature and pressure "upstream" known. In this case Potto–GDC provides the following table

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
5.5346	0.37554	0.0	1.989	5.479	34.50	0.021717

The calculations were carried as following: First calculate the M_x as

$$Mx = U_s / \sqrt{k * 287. * T_x}$$

Then calculate the M_y by using Potto-GDC or utilize the Tables. For example Potto-GDC (this code was produce by the program)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
5.5346	0.37554	5.4789	6.2963	34.4968	0.02172

The calculation of the temperature and pressure ratio also can be obtain by the same manner. The "downstream" shock number is

$$M_{sy} = \frac{U_s}{\sqrt{k * 287. * T_x * \left(\frac{T_y}{T_x}\right)}} \sim 2.09668$$

Finally utilizing the equation to calculate the following

$$M_y{}' = M_{sy} - M_y = 2.09668 - 0.41087 \sim 1.989$$

End Solution

Example 7.6:
An inventor interested in a design of tube and piston so that the pressure is doubled in the cylinder when the piston is moving suddenly. The propagating piston is assumed to move into media with temperature of $300K$ *and atmospheric pressure of* $1[Bar]$*. If the steady state is achieved, what will be the piston velocity?*

SOLUTION

This is an open valve case in which the pressure ratio is given. For this pressure ratio of $P_y/P_x = 2$ the following table can be obtained or by using Potto–GDC

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.3628	0.75593	1.2308	1.6250	2.0000	0.96697

The temperature ratio and the Mach numbers for the velocity of the air (and the piston) can be calculated. The temperature at "downstream" (close to the piston) is

$$T_y = T_x \frac{T_y}{T_x} = 300 \times 1.2308 = 369.24[^\circ\mathrm{C}]$$

The velocity of the piston is then

$$U_y = M_y * c_y = 0.75593 * \sqrt{1.4 * 287 * 369.24} \sim 291.16[m/sec]$$

End Solution

Example 7.7:
A flow of gas is brought into a sudden stop. The mass flow rate of the gas is 2 [kg/sec] and cross section A = $0.002[m^3]$*. The imaginary gas conditions are temperature is* $350K$ *and pressure is* $2[Bar]$ *and* $R = 143[j/kg\ K]$ *and* $k = 1.091$ *(Butane?). Calculate the conditions behind the shock wave.*

SOLUTION

This is the case of a closed valve in which mass flow rate with the area given. Thus, the "upstream" Mach is given.

$$U_x{}' = \frac{\dot{m}}{\rho A} = \frac{\dot{m}RT}{PA} = \frac{2 \times 287 \times 350}{200000 \times 0.002} \sim 502.25[m/sec]$$

Thus the static Mach number, $M_x{}'$ is

$$M_x{}' = \frac{U_x{}'}{c_x} = \frac{502.25}{\sqrt{1.091 \times 143 \times 350}} \sim 2.15$$

With this value for the Mach number Potto-GDC provides

$\mathbf{M_x}$	$\mathbf{M_y}$	$\mathbf{M_x}'$	$\mathbf{M_y}'$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{Po_y}}{\mathbf{Po_x}}$
2.9222	0.47996	2.1500	0.0	2.589	9.796	0.35101

This table was obtained by using the procedure described in this book. The iteration of the procedure are

i	$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\mathbf{M_y}'$
0	3.1500	0.46689	2.8598	11.4096	0.0
1	2.940	0.47886	2.609	9.914	0.0
2	2.923	0.47988	2.590	9.804	0.0
3	2.922	0.47995	2.589	9.796	0.0
4	2.922	0.47996	2.589	9.796	0.0
5	2.922	0.47996	2.589	9.796	0.0

End Solution

Example 7.8:
A converging–diverging nozzle is attached to reservoir with pressure of 10 [Bar] and temperature 300K. The air is released to the atmosphere which is about 1[Bar]. Assume that the flow is isentropic and shock–less. Calculate the velocity at the throat. Calculate the required throat area if the mass flow rate is 10[kg/sec]. It was measured that the nozzle efficiency is 90%. In that case, calculate the actual required exit area (assume that the flow in steady state and shock–free). The loss occurs in the converging section can be assumed to be zero and the only loss occurs in the diverging section. Assume that the spesifc heat ratio, k=1.45, and pressure heat capacity, $C_p = 1.04[kJ/kg^\circ\mathrm{C}]$.

SOLUTION

The velocity at the throat is the critical velocity hence the star conditions (Potto–GDC for k=1.45) are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
1.0000	0.81633	0.63700	1.0000	0.52000	0.52000	0.52000

The temperature at the throat is

$$T^* = T_0 \frac{T^*}{T_0} = 300 \times 0.81633 = 244.899K$$

The pressure at the throat is

$$P^* = P_0 \frac{P^*}{P_0} = 10 \times 0.52000 = 5.2[Bar]$$

The critical density, ρ^*, is

$$\rho^* = \frac{P^*}{R\,T^*} = \frac{520000}{287 \times 244.899} \sim 7.4[kg/m^3]$$

The throat area is obtained form the continuity equation as

$$A^* = \frac{\dot{m}}{\rho^* \sqrt{k\,R\,T^*}} \sim \frac{10}{7.4 \times \sqrt{1.4 \times 287 \times 244.899}} = 0.0043[m^2]$$

The question of efficiency requires knowledge of the isentropic temperature at the exit which can be obtained from the pressure ratio supersonic flow (using Potto–GDC) as

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
2.1534	0.48939	0.20434	1.8697	0.10000	0.18697	0.58945

It has to be noted that the above values are for the isentropic case and not the actual velocity. The actual exit temperature can be obtained from the definition of the efficiency as

$$\begin{aligned} \eta = \frac{T_0 - T_{exit}}{T_0 - T_s} \Longrightarrow T_{exit} = T_0 - \eta\,(T_0 - T_s) \\ \sim 300 - 0.9 \times (300 - 0.48939 \times 300) \sim 162.14K \end{aligned} \tag{7.VIII.a}$$

The energy equation can be utilized along with the actual exit temperature

$$\frac{\cancelto{=0}{U_0^2}}{2} - \frac{{U_{exit}}^2}{2} = C_p\,(T_{exit} - T_0) \tag{7.9}$$

The exit velocity is then

$$U_{exit} = \sqrt{2\,C_p\,(T_0 - T_{exit})} \sim \sqrt{2 * 1040\,(300 - 162.14)} \sim 535.48$$

The actual exit Mach number is

$$M = \frac{535.48}{1.4 \times 287 \times 162.14} \sim 2.09$$

It is interesting to point out that when the efficiency reduced in the Mach number reduced as well. The exit area obtain the continuity equation as

$$A_{exit} = \frac{\dot{m}}{\rho\, U_{exit}} = \frac{10}{7.4 \times 535.48} = 0.0025[m^2]$$

End Solution

Example 7.9:

Engineer designed a nozzle assuming ideal conditions to obtain a known flow rate of 4 [kg/s] when the upstream reservoir conditions are known and $20[Bar]$ and 27° C. The exit pressure is given (and for part a) 1[Bar].

(a) Calculate under these conditions what are Mach number, temperature, pressure at the throat and exit.

(b) After the operation, the measurements were made to find that the achieved the same flow rate 4[kg/s] but the correct back pressure for shock–less supersonic flow was 2.0 [Bar]. Calculate the nozzle efficiency. What is the required exit so the expansion will be as in the initial design (1[Bar]). For air take $R = 287 J/kgK$, and $k = 1.4$.

SOLUTION

(a) The speed of sound at the throat (equation (5.14)) is

$$c^* = \sqrt{k\, R\, \underbrace{\frac{2\, T_0}{k+2}}_{\rho^*}} = \sqrt{1.4 \times 287 \times \frac{2 \times 300}{1+1.4}} = 293.4[m/sec]$$

The star (throat) density is then

$$\rho^* = \rho_0 \frac{\rho^*}{\rho_0} = \frac{P_0}{R\, T_0} \left(\frac{2}{k+1}\right)^{\frac{1}{k-1}} = \frac{2000000}{287 \times 300} \times 0.633938 = 14.72[kg/m^3]$$

The throat area has to be

$$A^* = \frac{\dot{m}}{\rho^*\, c^*} = \frac{4}{14.72 \times 293.4} = 0.0009258[m^2]$$

The isentropic conditions (100%) efficiency between if the flow are determined by pressure ratio (obtained with Potto–GDC) also can be used.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
2.6015	0.42489	0.11768	2.9000	0.05000	0.14500	0.63286

The exit area is

$$A_{exit} = A^* * \frac{A_{exit}}{A^*} = 0.0009258 \times 2.9000 \sim 0.00268482[m^2]$$

The ideal exit temperatuer is

$$T_{exit} = 300 \times 0.42489 \sim 127.5K$$

(b) The second part; the flow rate remains the same but the exit pressure increase to $2[Bar]$. The mass flow rate is remained constant hence

$$\dot{m} = \rho_{exit}\, A_{exit}\, M_{exit}\, c_{exit} = \frac{\overbrace{P_{exit}}^{\rho_{exit}}}{R\, T_{exit}}\, A_{exit}\, M_{exit}\, \sqrt{k\, R\, T_{exit}} \tag{7.IX.a}$$

Rearranging equation (7.IX.a) results in

$$\dot{m} = \frac{P_{exit}\, A_{exit}\, M_{exit}\, \sqrt{k}}{\sqrt{R\, T_{exit}}} \tag{7.IX.b}$$

The energy equation can be used as following

$$\frac{T_0}{T_{exit}} = 1 + \frac{k-1}{2}\, {M_{exit}}^2 \tag{7.IX.c}$$

There are two unknowns, T_{exit} and M_{exit}, and two equations this system can be solved. The definition of the Fliegner's number (see page 99 equation (5.52).) relates the flow rate with the stagnation pressure and temperature[3]. Naturally equation (7.IX.a) can be written as

$$M_{exit} = \frac{\dot{m}\, \sqrt{R\, T_{exit}}}{P_{exit}\, A_{exit}\, \sqrt{k}} = \frac{\dot{m}\, \overbrace{\sqrt{k\, R\, T_0}}^{c_0}}{P_0\, A^*}\, \frac{1}{k}\, \sqrt{\frac{T_{exit}}{T_0}}\, \frac{A^*}{A_{exit}}\, \frac{P_0}{P_{exit}} \tag{7.IX.d}$$

or with regular definitions of the bar parameters as

$$\begin{aligned} M_{exit} = \frac{\boldsymbol{Fn}}{k}\, \sqrt{\frac{T_{exit}}{T_0}}\, \frac{A^*}{A_{exit}}\, \frac{P_0}{P_{exit}} = \quad & \frac{\boldsymbol{Fn}\, \sqrt{\overline{T}}}{k\, \overline{P}\, \overline{A}} \\ \text{or} \qquad \overline{T} = \quad & \left(\frac{k\, \overline{P}\, \overline{A}}{\boldsymbol{Fn}}\right)^2 {M_{exit}}^2 \end{aligned} \tag{7.IX.e}$$

[3]The Fliegner number, $\boldsymbol{Fn}$, is used because observation of the definition shows that

$$\boldsymbol{Fn} \equiv \frac{\dot{m}\, c_0}{A^*\, P_0} \equiv \frac{\dot{m}\, \sqrt{k\, R\, T_0}}{A^*\, P_0}$$

is constant for all choked flow.

Combining equation (7.IX.c) and (7.IX.e) results in

$$1+\frac{k-1}{2}{M_{exit}}^2=\left(\frac{\boldsymbol{Fn}}{k\,\overline{P}\,\overline{A}}\right)^2\frac{1}{{M_{exit}}^2}$$
$$\text{or}\qquad \frac{k-1}{2}{M_{exit}}^4+{M_{exit}}^2-\left(\frac{\boldsymbol{Fn}}{k\,\overline{P}\,\overline{A}}\right)^2=0 \tag{7.IX.f}$$

Equation (7.IX.f) in a quadratic equation for ${M_{exit}}^2$ and the solution is

$${M_{exit}}^2=\frac{-1\pm\sqrt{1+2\,(k-1)\left(\frac{\boldsymbol{Fn}}{k\,\overline{P}\,\overline{A}}\right)^2}}{k-1} \tag{7.IX.g}$$

The exit Mach number must be positive and hence must occur only with the positive sign a

$$M_{exit}=\sqrt{\frac{-1+\sqrt{1+\frac{2\,(k-1)}{k^2}\left(\frac{\boldsymbol{Fn}}{\overline{P}\,\overline{A}}\right)^2}}{k-1}} \tag{7.IX.h}$$

Figure 7.5 exhibits three dimensionless parameters which controlling the problem. It

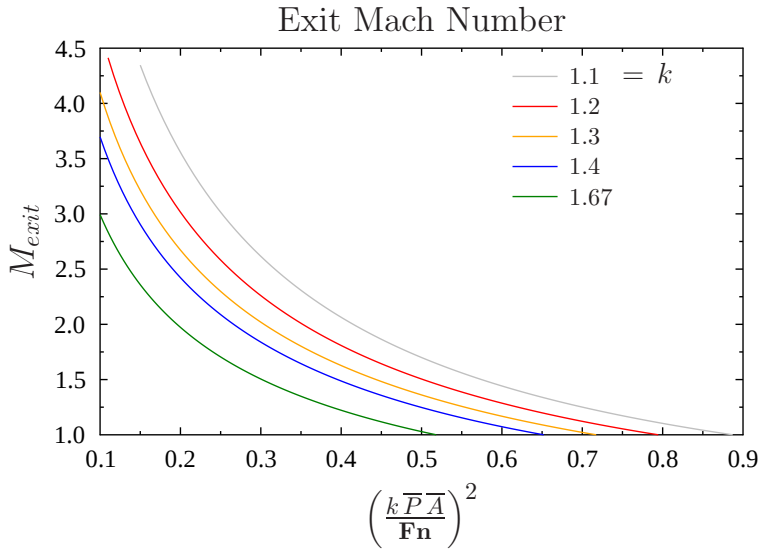


Fig. -7.5. Exit Mach number for non–ideal expansion in a converging–diverging nozzle general solution. The minimum exit number is one (1) since the solution refers to supersonic flow.

must be noted that exit Mach number is limited by one (1). When the flow become

subsonic, ($M < 1$), there must be a shock if the flow was choked at the throat. The efficiency is not due to construction of the nuzzle but rather the operation of the nuzzle. Hence, definition of nuzzle efficiency does not involve situation with shocks. On the other hand, the maximum Mach number is limited by the second law. The knowledge of these parameters yield the solution. The solution for the exit temperature, T_{exit} is obtained by utilizing the same governing equations. From equation (7.IX.e) can be transformed into Substituting into equation (7.IX.c)

$$\frac{1}{\overline{T}} = 1 + \frac{k-1}{2}\left(\frac{\boldsymbol{Fn}}{k\,\overline{P}\,\overline{A}}\right)^2 \overline{T} \tag{7.IX.i}$$

or after rearrangement equation (7.IX.i) becomes

$$\frac{k-1}{2}\left(\frac{\boldsymbol{Fn}}{k\,\overline{P}\,\overline{A}}\right)^2 \overline{T}^2 + \overline{T} - 1 = 0 \tag{7.IX.j}$$

The solution of equation (7.IX.j) is

$$\overline{T} = \frac{-1 \pm \sqrt{1 + 2\,(k-1)\left(\frac{\boldsymbol{Fn}}{k\,\overline{P}\,\overline{A}}\right)^2}}{(k-1)\left(\frac{\boldsymbol{Fn}}{k\,\overline{P}\,\overline{A}}\right)^2} \tag{7.IX.k}$$

Since the reltaive temperature, $\overline{T}$ is only possitive hence,

$$\overline{T} = \frac{-1 + \sqrt{1 + 2\,(k-1)\left(\frac{\boldsymbol{Fn}}{k\,\overline{P}\,\overline{A}}\right)^2}}{(k-1)\left(\frac{\boldsymbol{Fn}}{k\,\overline{P}\,\overline{A}}\right)^2} \tag{7.IX.l}$$

The stagnation speed of sound is

$$c_0 = \sqrt{k\,R\,T_0} = \sqrt{1.4 \times 287 \times 300} \sim 347.2[m/sec]$$

The Fliegner number ratio to pressure and area ratio is

$$\frac{\boldsymbol{Fn}}{A_{exit}\,P_{exit}} = \frac{4 \times 347.2}{0.00268482 \times 200000} \sim 2.59$$

Either using using Figure 7.6 as 0.68 (blue line) or equation (7.IX.l) provides the solution as

$$\overline{T_{exit}} = \frac{\sqrt{1 + \frac{2 \times 0.4}{1.4^2}\,(5.57)^2} - 1}{\frac{0.4}{1.4^2} \times (5.57)^2} \sim 0.682$$

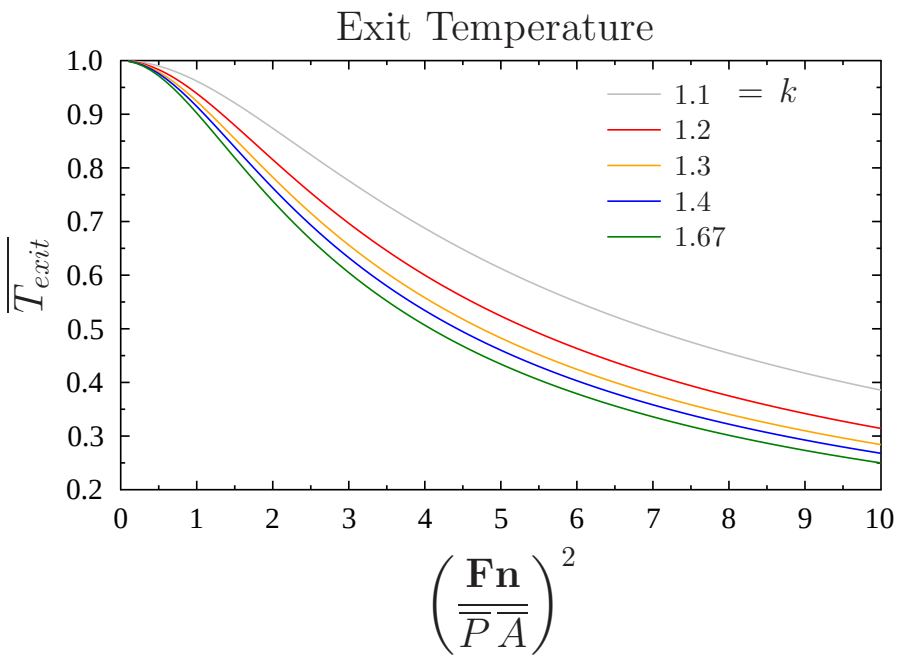


Fig. -7.6. The ratio of the exit temperature and stagnation temperature to be used for efficiency calculations.

Hence the exit temperature is

$$T_{exit} = T_0 \frac{T_{exit}}{T_0} \sim 300 \times 0.682 \sim 204.679K$$

The new efficiency is then

$$\eta = \frac{300 - 204.679}{300 - 127.5} = 0.552585507$$

End Solution

CHAPTER 8

Nozzle Flow With External Forces

This chapter is under heavy construction. Please ignore. If you want to contribute and add any results of experiments, to this chapter, please do so. You can help especially if you have photos showing these effects.

In the previous chapters a simple model describing the flow in nozzle was explained. In cases where more refined calculations have to carried the gravity or other forces have to be taken into account. Flow in a vertical or horizontal nozzle are different because the gravity. The simplified models that suggests them–self are: friction and adiabatic, isothermal, seem the most applicable. These models can served as limiting cases for more realistic flow.

The effects of the gravity of the nozzle flow in two models isentropic and isothermal is analyzed here. The isothermal nozzle model is suitable in cases where the flow is relatively slow (small Eckert numbers) while as the isentropic model is more suitable for large Eckert numbers.

The two models produces slightly different equations. The equations results in slightly different conditions for the choking and different choking speed. Moreover, the working equations are also different and this author isn't aware of material in the literature which provides any working table for the gravity effect.

8.1 Isentropic Nozzle ($Q = 0$)

The energy equation for isentropic nozzle provides

$$dh + UdU = \overbrace{f(x)dx}^{\substack{\text{external work or potential difference, i.e. } z \times g}} \tag{8.1}$$

Utilizing equation (5.26) when $ds = 0$ leads to

$$\frac{dP}{\rho} + UdU = f(x')dx' \tag{8.2}$$

For the isentropic process $dP = const \times k\rho^{k-1}d\rho$ when the $const = P/\rho^k$ at any point of the flow. The equation (8.2) becomes

$$\begin{aligned} \overbrace{\overbrace{\frac{P}{\rho^k}}^{any\ point} k\frac{\rho^k}{\rho}d\rho}^{dP}\frac{1}{\rho} + UdU &= k\overbrace{\frac{P}{\rho}}^{RT}\frac{d\rho}{\rho}UdU = f(x')dx' \\ \frac{kRTd\rho}{\rho} + UdU &= \frac{c^2}{\rho}d\rho + UdU = f(x')dx' \end{aligned} \tag{8.3}$$

The continuity equation as developed earlier (mass conservation equation isn't effected by the gravity)

$$-\frac{d\rho}{\rho} = \frac{dA}{A} + \frac{dU}{U} = 0 \tag{8.4}$$

Substituting $d\rho/\rho$ from equation 8.3, into equation (8.2) moving $d\rho$ to the right hand side, and diving by dx' yields

$$U\frac{dU}{dx'} = c^2\left[\frac{1}{U}\frac{dU}{dx'} + \frac{1}{A}\frac{dA}{dx'}\right] + f(x') \tag{8.5}$$

Rearranging equation (8.5) yields

$$\frac{dU}{dx'} = \left[M^2\frac{dU}{dx'} + \frac{c^2}{AU}\frac{dA}{dx'}\right] + \frac{f(x')}{U} \tag{8.6}$$

And further rearranging yields

$$\left(1 - M^2\right)\frac{dU}{dx'} = \frac{c^2}{AU}\frac{dA}{dx'} + \frac{f(x')}{U} \tag{8.7}$$

Equation (8.7) can be rearranged as

$$\frac{dU}{dx'} = \frac{1}{(1-M^2)}\left[\frac{c^2}{AU}\frac{dA}{dx'} + \frac{f(x')}{U}\right] \tag{8.8}$$

Equation (8.8) dimensionless form by utilizing $x = x'/\ell$ and ℓ is the nozzle length

$$\frac{dM}{dx} = \frac{1}{(1-M^2)}\left[\frac{1}{AM}\frac{dA}{dx} + \frac{\ell f(x)}{c\underbrace{cM}_{U}}\right] \tag{8.9}$$

And the final form of equation (8.9) is

Choking Flow Conditions

$$\frac{d\left(M^2\right)}{dx} = \frac{2}{(1-M^2)}\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] \tag{8.10}$$

The term $\frac{\ell f(x)}{c^2}$ is considered to be very small ($0.1 \times 10/100000 < 0.1\%$) for "standard" situations. The dimensionless number, $\frac{\ell f(x)}{c^2}$ sometimes referred as Ozer number determines whether gravity should be considered in the calculations. Nevertheless, one should be aware of value of Ozer number for large magnetic fields (astronomy) and low temperature, In such cases, the gravity effect can be considerable.

As it was shown before the transition must occur when $M = 1$. Consequently, two zones must be treated separately. First, here the Mach number is discussed and not the pressure as in the previous chapter. For $M < 1$ (the subsonic branch) the term $\frac{2}{(1-M^2)}$ is positive and the treads determined by gravity and the area function.

$$\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] > 0 \Longrightarrow d(M^2) > 0$$

or conversely,

$$\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] < 0 \Longrightarrow d(M^2) < 0$$

For the case of $M > 1$ (the supersonic branch) the term $\frac{2}{(1-M^2)}$ is negative and therefore

$$\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] > 0 \Longrightarrow d(M^2) < 0$$

For the border case $M = 1$, the denominator $1 - M^2 = 0$, is zero either $d(M^2) = \infty$ or

$$\left[\frac{1}{A}\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] = 0.$$

And the dM is indeterminate. As it was shown in chapter (5) the flow is chocked ($M = 1$) only when

$$\left[\frac{dA}{dx} + \frac{\ell f(x)}{c^2}\right] = 0. \tag{8.11}$$

It should be noticed that when $f(x)$ is zero, e.g. horizontal flow, the equation (8.11) reduced into $\frac{dA}{dx} = 0$ that was developed previously.

The ability to manipulate the location provides a mean to increase/decrease the flow rate. Yet this ability since Ozer number is relatively very small.

This condition means that the critical point can occurs in several locations that satisfies equation (8.11). Further, the critical point, sonic point is $\frac{dA}{Ax} \neq 0$ If $f(x)$ is a positive function, the critical point happen at converging part of the nozzle (before the throat) and if $f(x)$ is a negative function the critical point is diverging part of the throat. For example consider the gravity, $f(x) = -g$ a flow in a nozzle vertically the critical point will be above the throat.

8.2 Isothermal Nozzle $(T = constant)$

CHAPTER 9

Isothermal Flow

In this chapter a model dealing with gas that flows through a long tube is described. This model has a applicability to situations which occur in a relatively long distance and where heat transfer is relatively rapid so that the temperature can be treated, for engineering purposes, as a constant. For example, this model is applicable when a natural gas flows over several hundreds of meters. Such situations are common in large cities in U.S.A. where natural gas is used for heating. It is more predominant (more applicable) in situations where the gas is pumped over a length of kilometers.

The high speed of the gas is obtained or explained by the combination of heat transfer and the friction to the flow. For a long pipe, the pressure difference reduces the density of the gas. For instance, in a perfect gas, the density is inverse of the pressure (it has to be kept in mind that the gas undergoes an isothermal process.). To maintain conservation of mass, the velocity increases inversely to the pressure. At critical point the velocity reaches the speed of sound at the exit and hence the flow will be choked[1].

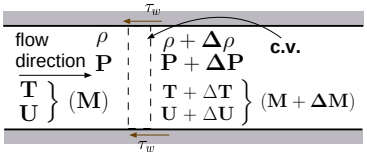


Fig. -9.1. Control volume for isothermal flow.

[1]This explanation is not correct as it will be shown later on. Close to the critical point (about, $1/\sqrt{k}$, the heat transfer, is relatively high and the isothermal flow model is not valid anymore. Therefore, the study of the isothermal flow above this point is only an academic discussion but also provides the upper limit for Fanno Flow.

9.1 The Control Volume Analysis/Governing equations

Figure (9.1) describes the flow of gas from the left to the right. The heat transfer up stream (or down stream) is assumed to be negligible. Hence, the energy equation can be written as the following:

$$\frac{dQ}{\dot{m}} = c_p dT + d\frac{U^2}{2} = c_p dT_0 \tag{9.1}$$

The momentum equation is written as the following

$$-AdP - \tau_w dA_{\text{wetted area}} = \dot{m} dU \tag{9.2}$$

where A is the cross section area (it doesn't have to be a perfect circle; a close enough shape is sufficient.). The shear stress is the force per area that acts on the fluid by the tube wall. The $A_{wetted\ area}$ is the area that shear stress acts on. The second law of thermodynamics reads

$$\frac{s_2 - s_1}{C_p} = \ln\frac{T_2}{T_1} - \frac{k-1}{k}\ln\frac{P_2}{P_1} \tag{9.3}$$

The mass conservation is reduced to

$$\dot{m} = \text{constant} = \rho\, U\, A \tag{9.4}$$

Again it is assumed that the gas is a perfect gas and therefore, equation of state is expressed as the following:

$$P = \rho\, R\, T \tag{9.5}$$

9.2 Dimensionless Representation

In this section the equations are transformed into the dimensionless form and presented as such. First it must be recalled that the temperature is constant and therefore, equation of state reads

$$\frac{dP}{P} = \frac{d\rho}{\rho} \tag{9.6}$$

It is convenient to define a hydraulic diameter

$$D_H = \frac{4 \times \text{Cross Section Area}}{\text{wetted perimeter}} \tag{9.7}$$

The Fanning friction factor[2] is introduced, this factor is a dimensionless friction factor sometimes referred to as the friction coefficient as

$$f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{9.8}$$

[2]It should be noted that Fanning factor based on hydraulic radius, instead of diameter friction equation, thus "Fanning f" values are only 1/4th of "Darcy f" values.

Substituting equation (9.8) into momentum equation (9.2) yields

$$-dP - \frac{4dx}{D_H} f\left(\frac{1}{2}\rho U^2\right) = \overbrace{\rho U}^{\frac{\dot{m}}{A}} dU \tag{9.9}$$

Rearranging equation (9.9) and using the identify for perfect gas $M^2 = \rho U^2/kP$ yields:

$$-\frac{dP}{P} - \frac{4f dx}{D_H}\left(\frac{k\,P\,M^2}{2}\right) = \frac{k\,P\,M^2 dU}{U} \tag{9.10}$$

The pressure, P as a function of the Mach number has to substitute along with velocity, U as

$$U^2 = k\,R\,T M^2 \tag{9.11}$$

Differentiation of equation (9.11) yields

$$d(U^2) = k\,R\,\left(M^2\,dT + T\,d(M^2)\,\right) \tag{9.12}$$

$$\frac{d(M^2)}{M^2} = \frac{d(U^2)}{U^2} - \frac{dT}{T} \tag{9.13}$$

It can be noticed that $dT = 0$ for isothermal process and therefore

$$\frac{d(M^2)}{M^2} = \frac{d(U^2)}{U^2} = \frac{2U\,dU}{U^2} = \frac{2dU}{U} \tag{9.14}$$

The dimensionalization of the mass conservation equation yields

$$\frac{d\rho}{\rho} + \frac{dU}{U} = \frac{d\rho}{\rho} + \frac{2\,U\,dU}{2U^2} = \frac{d\rho}{\rho} + \frac{d(U^2)}{2\,U^2} = 0 \tag{9.15}$$

Differentiation of the isotropic (stagnation) relationship of the pressure (5.11) yields

$$\frac{dP_0}{P_0} = \frac{dP}{P} + \left(\frac{\frac{k\,M^2}{2}}{1 + \frac{k-1}{2}M^2}\right)\,\frac{dM^2}{M^2} \tag{9.16}$$

Differentiation of equation (5.9) yields:

$$dT_0 = dT\left(1 + \frac{k-1}{2}M^2\right) + T\,\frac{k-1}{2}\,dM^2 \tag{9.17}$$

Notice that $dT_0 \neq 0$ in an isothermal flow. There is no change in the actual temperature of the flow but the stagnation temperature increases or decreases depending

on the Mach number (supersonic flow of subsonic flow). Substituting T for equation (9.17) yields:

$$dT_0 = \frac{T_0 \, \dfrac{k-1}{2} \, d\,M^2}{\left(1 + \dfrac{k-1}{2} M^2\right)} \, \frac{M^2}{M^2} \tag{9.18}$$

Rearranging equation (9.18) yields

$$\frac{dT_0}{T_0} = \frac{(k-1)\, M^2}{2\left(1 + \dfrac{k-1}{2}\right)} \, \frac{dM^2}{M^2} \tag{9.19}$$

By utilizing the momentum equation it is possible to obtain a relation between the pressure and density. Recalling that an isothermal flow ($dT = 0$) and combining it with perfect gas model yields

$$\frac{dP}{P} = \frac{d\rho}{\rho} \tag{9.20}$$

From the continuity equation (see equation (9.14)) leads

$$\frac{dM^2}{M^2} = \frac{2dU}{U} \tag{9.21}$$

The four equations momentum, continuity (mass), energy, state are described above. There are 4 unknowns (M, T, P, ρ)[3] and with these four equations the solution is attainable. One can notice that there are two possible solutions (because of the square power). These different solutions are supersonic and subsonic solution.

The distance friction, $\frac{4\,f\,L}{D}$, is selected as the choice for the independent variable. Thus, the equations need to be obtained as a function of $\frac{4f\,L}{D}$. The density is eliminated from equation (9.15) when combined with equation (9.20) to become

$$\frac{dP}{P} = -\frac{dU}{U} \tag{9.22}$$

After substituting the velocity (9.22) into equation (9.10), one can obtain

$$-\frac{dP}{P} - \frac{4f dx}{D_H}\left(\frac{k\,P\,M^2}{2}\right) = k\,P\,M^2 \frac{dP}{P} \tag{9.23}$$

Equation (9.23) can be rearranged into

$$\frac{dP}{P} = \frac{d\rho}{\rho} = -\frac{dU}{U} = -\frac{1}{2}\frac{dM^2}{M^2} = -\frac{k\,M^2}{2\,(1 - k\,M^2)} 4\,f \frac{dx}{D} \tag{9.24}$$

[3]Assuming the upstream variables are known.

Similarly or by other paths, the stagnation pressure can be expressed as a function of $\frac{4\,f\,L}{D}$

$$\frac{dP_0}{P_0} = \frac{k\,M^2\left(1-\frac{k+1}{2}\,M^2\right)}{2\,(k\,M^2-1)\left(1+\frac{k-1}{2}M^2\right)}4\,f\,\frac{dx}{D} \tag{9.25}$$

$$\frac{dT_0}{T_0} = \frac{k\,(1-k)\,M^2}{2\,(1-k\,M^2)\left(1+\frac{k-1}{2}M^2\right)}\,4\,f\frac{dx}{D} \tag{9.26}$$

The variables in equation (9.24) can be separated to obtain integrable form as follows

$$\int_0^L \frac{4f dx}{D} = \int_{M^2}^{1/k} \frac{1-k\,M2}{k\,M2} dM^2 \tag{9.27}$$

It can be noticed that at the entrance $(x=0)$ for which $M = M_{x=0}$ (the initial velocity in the tube isn't zero). The term $\frac{4f\,L}{D}$ is positive for any x, thus, the term on the other side has to be positive as well. To obtain this restriction $1 = k\,M^2$. Thus, the value $M = \frac{1}{\sqrt{k}}$ is the limiting case from a mathematical point of view. When Mach number larger than $M > \frac{1}{\sqrt{k}}$ it makes the right hand side of the integrate negative. The physical meaning of this value is similar to $M = 1$ choked flow which was discussed in a variable area flow in Chapter (5).

Further it can be noticed from equation (9.26) that when $M \to \frac{1}{\sqrt{k}}$ the value of right hand side approaches infinity (∞). Since the stagnation temperature (T_0) has a finite value which means that $dT_0 \to \infty$. Heat transfer has a limited value therefore the model of the flow must be changed. A more appropriate model is an adiabatic flow model yet this model can serve as a bounding boundary (or limit).

Integration of equation (9.27) requires information about the relationship between the length, x, and friction factor f. The friction is a function of the Reynolds number along the tube. Knowing the Reynolds number variations is important. The Reynolds number is defined as

$$Re = \frac{D\,U\,\rho}{\mu} \tag{9.28}$$

The quantity $U\,\rho$ is constant along the tube (mass conservation) under constant area. Thus, only viscosity is varied along the tube. However under the assumption of ideal gas, viscosity is only a function of the temperature. The temperature in isothermal process (the definition) is constant and thus the viscosity is constant. In real gas, the pressure effects are very minimal as described in "Basic of fluid mechanics" by this

author. Thus, the friction factor can be integrated to yield

Friction Mach

$$\left.\frac{4\,f\,L}{D}\right|_{max} = \frac{1-k\,M^2}{k\,M^2} + \ln\left(k\,M^2\right) \tag{9.29}$$

The definition for perfect gas yields $M^2 = U^2/k\,RT$ and noticing that $T = constant$ is used to describe the relation of the properties at $M = 1/\sqrt{k}$. By denoting the superscript symbol $*$ for the choking condition, one can obtain that

$$\frac{M^2}{U^2} = \frac{1/k}{U^{*2}} \tag{9.30}$$

Rearranging equation (9.30) is transformed into

$$\frac{U}{U^*} = \sqrt{k}M \tag{9.31}$$

Utilizing the continuity equation provides

$$\rho U = \rho^* U^*; \Longrightarrow \frac{\rho}{\rho^*} = \frac{1}{\sqrt{k}\,M} \tag{9.32}$$

Reusing the perfect–gas relationship

Pressure Ratio

$$\frac{P}{P^*} = \frac{\rho}{\rho^*} = \frac{1}{\sqrt{k}\,M} \tag{9.33}$$

Utilizing the relation for stagnated isotropic pressure one can obtain

$$\frac{P_0}{P_0^*} = \frac{P}{P^*}\left[\frac{1+\frac{k-1}{2}M^2}{1+\frac{k-1}{2k}}\right]^{\frac{k}{k-1}} \tag{9.34}$$

Substituting for $\frac{P}{P^*}$ equation (9.33) and rearranging yields

Stagnation Pressure Ratio

$$\frac{P_0}{P_0^*} = \frac{1}{\sqrt{k}}\left(\frac{2\,k}{3\,k-1}\right)^{\frac{k}{k-1}}\left(1+\frac{k-1}{2}M^2\right)^{\frac{k}{k-1}}\frac{1}{M} \tag{9.35}$$

And the stagnation temperature at the critical point can be expressed as

Stagnation Pressure Ratio

$$\frac{T_0}{T_0^*} = \frac{T}{T^*}\frac{1+\dfrac{k-1}{2}M^2}{1+\dfrac{k-1}{2\,k}} = \frac{2\,k}{3k-1}\left(1+\frac{k-1}{2}\right)M^2 \tag{9.36}$$

These equations (9.31)-(9.36) are presented on in Figure (9.2).

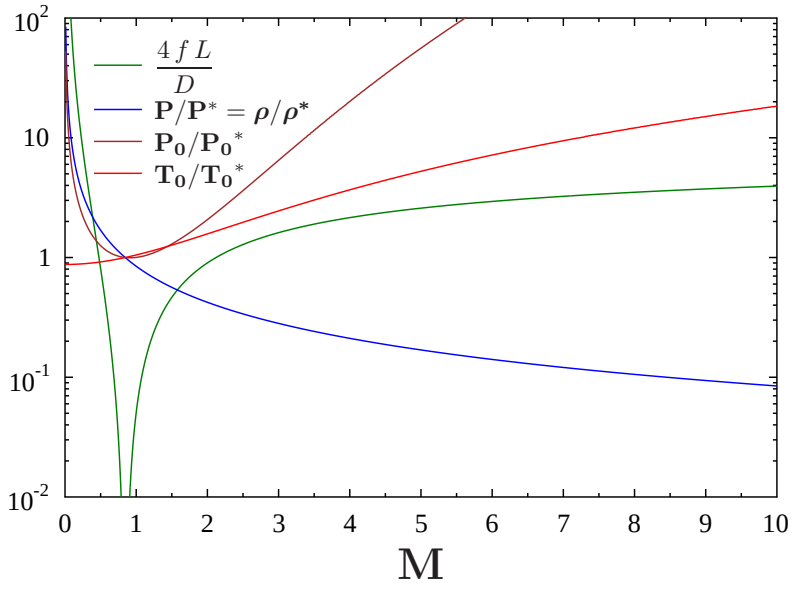


Fig. -9.2. Description of the pressure, temperature relationships as a function of the Mach number for isothermal flow.

9.3 The Entrance Limitation of Supersonic Branch

Situations where the conditions at the tube exit have not arrived at the critical conditions are discussed here. It is very useful to obtain the relationships between the entrance and the exit conditions for this case. Denote *1* and *2* as the conditions at the inlet and exit respectably. From equation (9.24)

$$\frac{4fL}{D} = \frac{4fL}{D}\bigg|_{max_1} - \frac{4fL}{D}\bigg|_{max_2} = \frac{1-kM_1{}^2}{kM_1{}^2} - \frac{1-kM_2{}^2}{kM_2{}^2} + \ln\left(\frac{M_1}{M_2}\right)^2 \quad (9.37)$$

For the case that $M_1 >> M_2$ and $M_1 \to 1$ equation (9.37) is reduced into the following approximation

$$\frac{4fL}{D} = 2\ln(M_1) - 1 - \overbrace{\frac{1-kM_2{}^2}{kM_2{}^2}}^{\sim 0} \quad (9.38)$$

Solving for M_1 results in

$$M_1 \sim \mathrm{e}^{\frac{1}{2}\left(\frac{4fL}{D}+1\right)} \quad (9.39)$$

This relationship shows the maximum limit that Mach number can approach when the heat transfer is extraordinarily fast. In reality, even small $\frac{4fL}{D} > 2$ results in a Mach number which is larger than 4.5. This velocity requires a large entrance length to achieve good heat transfer. With this conflicting mechanism obviously the flow is closer to the Fanno flow model. Yet this model provides the directions of the heat transfer effects on the flow.

Example 9.1:

Calculate the exit Mach number for pipe with $\frac{4fL}{D} = 3$ *under the assumption of the isothermal flow and supersonic flow. Estimate the heat transfer needed to achieve this flow.*

9.4 Comparison with Incompressible Flow

The Mach number of the flow in some instances is relatively small. In these cases, one should expect that the isothermal flow should have similar characteristics as incompressible flow. For incompressible flow, the pressure loss is expressed as follows

$$P_1 - P_2 = \frac{4fL}{D}\frac{U^2}{2} \tag{9.40}$$

Now note that for incompressible flow $U_1 = U_2 = U$ and $\frac{4fL}{D}$ represent the ratio of the traditional h_{12}. To obtain a similar expression for isothermal flow, a relationship between M_2 and M_1 and pressures has to be derived. From equation (9.40) one can obtained that

$$M_2 = M_1 \frac{P_1}{P_2} \tag{9.41}$$

Substituting this expression into (9.41) yields

$$\frac{4fL}{D} = \frac{1}{k\,{M_1}^2}\left(1 - \left(\frac{P_2}{P_1}\right)^2\right) - \ln\left(\frac{P_2}{P_1}\right)^2 \tag{9.42}$$

Because f is always positive there is only one solution to the above equation even though M2.

Expanding the solution for small pressure ratio drop, $P_1 - P_2/P_1$, by some mathematics. Denote

$$\chi = \frac{P_1 - P_2}{P_1} \tag{9.43}$$

Now equation (9.42) can be transformed into

$$\frac{4fL}{D} = \frac{1}{k\,{M_1}^2}\left(1 - \left(\frac{P_2 - P_1 + P_1}{P_1}\right)^2\right) - \ln\left(\frac{1}{\frac{P_2}{P_1}}\right)^2 \tag{9.44}$$

$$\frac{4\,f\,L}{D} = \frac{1}{k\,{M_1}^2}\left(1-(1-\chi)^2\right) - \ln\left(\frac{1}{1-\chi}\right)^2 \tag{9.45}$$

$$\frac{4\,f\,L}{D} = \frac{1}{k{M_1}^2}\left(2\chi-\chi^2\right) - \ln\left(\frac{1}{1-\chi}\right)^2 \tag{9.46}$$

now we have to expand into a series around $\chi = 0$ and remember that

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + 0\left(x^3\right) \tag{9.47}$$

and for example the first derivative of

$$\frac{d}{d\chi}\ln\left(\frac{1}{1-\chi}\right)^2\Bigg|_{\chi=0} =$$
$$(1-\chi)^2 \times \left[(-2)(1-\chi)^{-3}\right](-1)\Big|_{\chi=0} = \qquad 2 \tag{9.48}$$

similarly it can be shown that $f''(\chi = 0) = 1$ equation (9.46) now can be approximated as

$$\frac{4\,f\,L}{D} = \frac{1}{k{M_1}^2}(2\chi-\chi^2) - \left(2\chi-\chi^2\right) + f\left(\chi^3\right) \tag{9.49}$$

rearranging equation (9.49) yields

$$\frac{4\,f\,L}{D} = \frac{\chi}{k{M_1}^2}\left[(2-\chi) - k{M_1}^2\,(2-\chi)\right] + f\left(\chi^3\right) \tag{9.50}$$

and further rearrangement yields

$$\frac{4\,f\,L}{D} = \frac{\chi}{k{M_1}^2}\left[2(1-k{M_1}^2) - \left(1+k{M_1}^2\right)\chi\right] + f\left(\chi^3\right) \tag{9.51}$$

in cases that χ is small

$$\frac{4\,f\,L}{D} \approx \frac{\chi}{k\,{M_1}^2}\left[2(1-k\,{M_1}^2) - \left(1+k\,{M_1}^2\right)\chi\right] \tag{9.52}$$

The pressure difference can be plotted as a function of the M_1 for given value of $\frac{4\,f\,L}{D}$. Equation (9.52) can be solved explicitly to produce a solution for

$$\chi = \frac{1-k\,{M_1}^2}{1+k\,{M_1}^2} - \sqrt{\frac{1-k\,{M_1}^2}{1+k\,{M_1}^2} - \frac{k\,{M_1}^2}{1+k\,{M_1}^2}\frac{4\,f\,L}{D}} \tag{9.53}$$

A few observations can be made about equation (9.53).

9.5 Supersonic Branch

Apparently, this analysis/model is over simplified for the supersonic branch and does not produce reasonable results since it neglects to take into account the heat transfer effects. A dimensionless analysis[4] demonstrates that all the common materials that the author is familiar which creates a large error in the fundamental assumption of the model and the model breaks. Nevertheless, this model can provide a better understanding to the trends and deviations from Fanno flow model.

In the supersonic flow, the hydraulic entry length is very large as will be shown below. However, the feeding diverging nozzle somewhat reduces the required entry length (as opposed to converging feeding). The thermal entry length is in the order of the hydrodynamic entry length (look at the Prandtl number[5] , (0.7-1.0), value for the common gases.). Most of the heat transfer is hampered in the sublayer thus the core assumption of isothermal flow (not enough heat transfer so the temperature isn't constant) breaks down[6].

The flow speed at the entrance is very large, over hundred of meters per second. For example, a gas flows in a tube with $\frac{4fL}{D} = 10$ the required entry Mach number is over 200. Almost all the perfect gas model substances dealt with in this book, the speed of sound is a function of temperature. For this illustration, for most gas cases the speed of sound is about $300[m/sec]$. For example, even with low temperature like $200K$ the speed of sound of air is $283[m/sec]$. So, even for relatively small tubes with $\frac{4fD}{D} = 10$ the inlet speed is over 56 [km/sec]. This requires that the entrance length to be larger than the actual length of the tub for air. Remember from "Basics of Fluid Mechanics"[7]

$$L_{entrance} = 0.06 \frac{U D}{\nu} \tag{9.54}$$

The typical values of the the kinetic viscosity, ν, are 0.0000185 kg/m-sec at 300K and 0.0000130034 kg/m-sec at 200K. Combine this information with our case of $\frac{4fL}{D} = 10$

$$\frac{L_{entrance}}{D} = 250746268.7$$

On the other hand a typical value of friction coefficient $f = 0.005$ results in

$$\frac{L_{max}}{D} = \frac{10}{4 \times 0.005} = 500$$

The fact that the actual tube length is only less than 1% of the entry length means that the assumption is that the isothermal flow also breaks (as in a large response time). If Mach number is changing from 10 to 1 the kinetic energy change is about $\frac{\mathbf{T_0}}{\mathbf{T_0}^*} = 18.37$ which means that the maximum amount of energy is insufficient.

[4] This dimensional analysis is a bit tricky, and is based on estimates. Currently and ashamedly the author is looking for a more simplified explanation. The current explanation is correct but based on hands waving and definitely does not satisfy the author.

[5] is relating thermal boundary layer to the momentum boundary layer.

[6] See Kays and Crawford "Convective Heat Transfer" (equation 12-12).

[7] Basics of Fluid Mechanics, Bar-Meir, Genick, Potto Project, 2013

Now with limitation, this topic will be covered in the next version because it provide some insight and boundary to the Fanno Flow model.

9.6 Figures and Tables

Table -9.1. The Isothermal Flow basic parameters

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0{}^*}}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0{}^*}}$
0.03000	785.97	28.1718	17.6651	28.1718	0.87516
0.04000	439.33	21.1289	13.2553	21.1289	0.87528
0.05000	279.06	16.9031	10.6109	16.9031	0.87544
0.06000	192.12	14.0859	8.8493	14.0859	0.87563
0.07000	139.79	12.0736	7.5920	12.0736	0.87586
0.08000	105.89	10.5644	6.6500	10.5644	0.87612
0.09000	82.7040	9.3906	5.9181	9.3906	0.87642
0.10000	66.1599	8.4515	5.3334	8.4515	0.87675
0.20000	13.9747	4.2258	2.7230	4.2258	0.88200
0.25000	7.9925	3.3806	2.2126	3.3806	0.88594
0.30000	4.8650	2.8172	1.8791	2.8172	0.89075
0.35000	3.0677	2.4147	1.6470	2.4147	0.89644
0.40000	1.9682	2.1129	1.4784	2.1129	0.90300
0.45000	1.2668	1.8781	1.3524	1.8781	0.91044
0.50000	0.80732	1.6903	1.2565	1.6903	0.91875
0.55000	0.50207	1.5366	1.1827	1.5366	0.92794
0.60000	0.29895	1.4086	1.1259	1.4086	0.93800
0.65000	0.16552	1.3002	1.0823	1.3002	0.94894
0.70000	0.08085	1.2074	1.0495	1.2074	0.96075
0.75000	0.03095	1.1269	1.0255	1.1269	0.97344
0.80000	0.00626	1.056	1.009	1.056	0.98700
0.81000	0.00371	1.043	1.007	1.043	0.98982

Table -9.1. The Isothermal Flow basic parameters (continue)

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.81879	0.00205	1.032	1.005	1.032	0.99232
0.82758	0.000896	1.021	1.003	1.021	0.99485
0.83637	0.000220	1.011	1.001	1.011	0.99741
0.84515	0.0	1.000	1.000	1.000	1.000

9.7 Isothermal Flow Examples

There can be several kinds of questions aside from the proof questions[8]. Generally, the "engineering" or practical questions can be divided into driving force (pressure difference), resistance (diameter, friction factor, friction coefficient, etc.), and mass flow rate questions. In this model no questions about shock (should) exist[9].

The driving force questions deal with what should be the pressure difference to obtain certain flow rate. Here is an example.

Example 9.2:
A tube of 0.25 [m] diameter and 5000 [m] in length is attached to a pump. What should be the pump pressure so that a flow rate of 2 $[kg/sec]$ *will be achieved? Assume that friction factor* $f = 0.005$ *and the exit pressure is* $1[bar]$. *The specific heat for the gas,* $k = 1.31$, *surroundings temperature* 27°C, $R = 290\left[\frac{J}{Kkg}\right]$. Hint: calculate the maximum flow rate and then check if this request is reasonable.

SOLUTION

If the flow was incompressible then for known density, ρ, the velocity can be calculated by utilizing $\Delta P = \frac{4\,f\mathbf{L}}{D}\frac{U^2}{2g}$. In incompressible flow, the density is a function of the entrance Mach number. The exit Mach number is not necessarily $1/\sqrt{k}$ i.e. the flow is not choked. First, check whether flow is choked (or even possible).

Calculating the resistance, $\frac{4\,f\,L}{D}$

$$\frac{4\,f\,L}{D} = \frac{4 \times 0.0055000}{0.25} = 400$$

Utilizing Table (9.1) or the Potto–GDC provides

[8] The proof questions are questions that ask for proof or for finding a mathematical identity (normally good for mathematicians and study of perturbation methods). These questions or examples will appear in the later versions.

[9] Those who are mathematically inclined can include these kinds of questions but there are no real world applications to isothermal model with shock.

$\mathbf{M}$	$\frac{\mathbf{4\,f\,l}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0{}^*}}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0{}^*}}$
0.04331	400.00	20.1743	12.5921	0.0	0.89446

The maximum flow rate (the limiting case) can be calculated by utilizing the above table. The velocity of the gas at the entrance $U = cM = 0.04331 \times \sqrt{1.31 \times 290 \times 300} \cong 14.62 \left[\frac{m}{sec}\right]$. The density reads

$$\rho = \frac{P}{RT} = \frac{2,017,450}{290 \times 300} \cong 23.19 \left[\frac{kg}{m^3}\right]$$

The maximum flow rate then reads

$$\dot{m} = \rho A U = 23.19 \times \frac{\pi \times (0.25)^2}{4} \times 14.62 \cong 16.9 \left[\frac{kg}{sec}\right]$$

The maximum flow rate is larger then the requested mass rate hence the flow is not choked. It is note worthy to mention that since the isothermal model breaks around the choking point, the flow rate is really some what different. It is more appropriate to assume an isothermal model hence our model is appropriate.

To solve this problem the flow rate has to be calculated as

$$\dot{m} = \rho A U = 2.0 \left[\frac{kg}{sec}\right]$$

$$\dot{m} = \frac{P_1}{RT} A \frac{kU}{k} = \frac{P_1}{\sqrt{kRT}} A \frac{kU}{\sqrt{kRT}} = \frac{P_1}{c} A k M_1$$

Now combining with equation (9.41) yields

$$\dot{m} = \frac{M_2 P_2 A k}{c}$$

$$M_2 = \frac{\dot{m} c}{P_2 A k} = \frac{2 \times 337.59}{100000 \times \frac{\pi \times (0.25)^2}{4} \times 1.31} = 0.103$$

From Table (9.1) or by utilizing the Potto–GDC one can obtain

$\mathbf{M}$	$\frac{\mathbf{4\,f\,L}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0{}^*}}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0{}^*}}$
0.10300	66.6779	8.4826	5.3249	0.0	0.89567

The entrance Mach number is obtained by

$$\left.\frac{4\,f\,L}{D}\right|_1 = 66.6779 + 400 \cong 466.68$$

Hence,

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0^*}}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0^*}}$
0.04014	466.68	21.7678	13.5844	0.0	0.89442

The pressure should be

$$P = 21.76780 \times 8.4826 = 2.566[bar]$$

Note that tables in this example are for $k = 1.31$

End Solution

Example 9.3:
A flow of gas was considered for a distance of 0.5 [km] (500 [m]). A flow rate of 0.2 [kg/sec] is required. Due to safety concerns, the maximum pressure allowed for the gas is only 10[bar]. Assume that the flow is isothermal and k=1.4, calculate the required diameter of tube. The friction coefficient for the tube can be assumed as 0.02 (A relative smooth tube of cast iron.). Note that tubes are provided in increments of 0.5 [in][10]*. You can assume that the soundings temperature to be* 27°C.

SOLUTION

At first, the minimum diameter will be obtained when the flow is choked. Thus, the maximum M_1 that can be obtained when the M_2 is at its maximum and back pressure is at the atmospheric pressure.

$$M_1 = M_2 \frac{P_2}{P_1} = \overbrace{\frac{1}{\sqrt{k}}}^{M_{max}} \frac{1}{10} = 0.0845$$

Now, with the value of M_1 either by utilizing Table (9.1) or using the provided program yields

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0^*}}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0^*}}$
0.08450	94.4310	10.0018	6.2991	0.0	0.87625

With $\left.\frac{4fL}{D}\right|_{max} = 94.431$, the value of minimum diameter.

$$D = \frac{4fL}{\left.\frac{4fL}{D}\right|_{max}} \simeq \frac{4 \times 0.02 \times 500}{94.43} \simeq 0.42359[m] = 16.68[in]$$

[10] It is unfortunate, but it seems that this standard will be around in USA for some time.

However, the pipes are provided only in 0.5 increments and the next size is $17[in]$ or $0.4318[m]$. With this pipe size the calculations are to be repeated in reverse and produces: (Clearly the maximum mass is determined with)

$$\dot{m} = \rho A U = \rho A M c = \frac{P}{RT} A M \sqrt{k R T} = \frac{P A M \sqrt{k}}{\sqrt{RT}}$$

The usage of the above equation clearly applied to the whole pipe. The only point that must be emphasized is that all properties (like Mach number, pressure and etc) have to be taken at the same point. The new $\frac{4 f L}{D}$ is

$$\frac{4 f L}{D} = \frac{4 \times 0.02 \times 500}{0.4318} \simeq 92.64$$

$\mathbf{M}$	$\frac{\mathbf{4 f L}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$
0.08527	92.6400	9.9110	6.2424	0.0	0.87627

To check whether the flow rate satisfies the requirement

$$\dot{m} = \frac{10^6 \times \frac{\pi \times 0.4318^2}{4} \times 0.0853 \times \sqrt{1.4}}{\sqrt{287 \times 300}} \approx 50.3[kg/sec]$$

Since $50.3 \geq 0.2$ the mass flow rate requirement is satisfied.

It should be noted that P should be replaced by P_0 in the calculations. The speed of sound at the entrance is

$$c = \sqrt{k R T} = \sqrt{1.4 \times 287 \times 300} \cong 347.2 \left[\frac{m}{sec}\right]$$

and the density is

$$\rho = \frac{P}{RT} = \frac{1,000,000}{287 \times 300} = 11.61 \left[\frac{kg}{m^3}\right]$$

The velocity at the entrance should be

$$U = M * c = 0.08528 \times 347.2 \cong 29.6 \left[\frac{m}{sec}\right]$$

The diameter should be

$$D = \sqrt{\frac{4\dot{m}}{\pi U \rho}} = \sqrt{\frac{4 \times 0.2}{\pi \times 29.6 \times 11.61}} \cong 0.027$$

Nevertheless, for the sake of the exercise the other parameters will be calculated. This situation is reversed question. The flow rate is given with the diameter of the pipe. It should be noted that the flow isn't choked.

End Solution

Example 9.4:
A gas flows of from a station (a) with pressure of 20[bar] through a pipe with 0.4[m] diameter and 4000 [m] length to a different station (b). The pressure at the exit (station (b)) is 2[bar]. The gas and the sounding temperature can be assumed to be 300 K. Assume that the flow is isothermal, k=1.4, and the average friction f=0.01. Calculate the Mach number at the entrance to pipe and the flow rate.

SOLUTION

First, the information whether the flow is choked needs to be found. Therefore, at first it will be assumed that the whole length is the maximum length.

$$\left.\frac{4\,f\,L}{D}\right|_{max} = \frac{4 \times 0.01 \times 4000}{0.4} = 400$$

with $\left.\frac{4\,f\,L}{D}\right|_{max} = 400$ the following can be written

$\mathbf{M}$	$\frac{\mathbf{4\,f\,L}}{\mathbf{D}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^{*\mathbf{T}}}$	$\frac{\rho}{\rho^{*\mathbf{T}}}$	$\frac{\mathbf{P}}{\mathbf{P}^{*\mathbf{T}}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^{*\mathbf{T}}}$
0.0419	400.72021	0.87531	20.19235	20.19235	12.66915

From the table $M_1 \approx 0.0419$,and $\frac{P_0}{{P_0}^{*T}} \approx 12.67$

$$ {P_0}^{*T} \cong \frac{28}{12.67} \simeq 2.21[bar]$$

The pressure at point (b) by utilizing the isentropic relationship ($M = 1$) pressure ratio is 0.52828.

$$P_2 = \frac{{P_0}^{*T}}{\left(\frac{P_2}{{P_0}^{*T}}\right)} = 2.21 \times 0.52828 = 1.17[bar]$$

As the pressure at point (b) is smaller than the actual pressure $P^* < P_2$ than the actual pressure one must conclude that the flow is not choked. The solution is an iterative process.

1. guess reasonable value of M_1 and calculate $\frac{4\,f\,L}{D}$
2. Calculate the value of $\left.\frac{4\,f\,L}{D}\right|_2$ by subtracting $\left.\frac{4\,f\,L}{D}\right|_1 - \frac{4\,f\,L}{D}$
3. Obtain M_2 from the Table ? or by using the Potto–GDC.
4. Calculate the pressure, P_2 bear in mind that this isn't the real pressure but based on the assumption.

5. Compare the results of guessed pressure P_2 with the actual pressure and choose new Mach number M_1 accordingly.

The process has been done and is provided in Figure or in a table obtained from Potto–GDC.

$\mathbf{M_1}$	$\mathbf{M_2}$	$\frac{\mathbf{4fL}}{\mathbf{D}}\Big\vert_{\mathbf{max}}\Big\vert_{\mathbf{1}}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$
0.0419	0.59338	400.32131	400.00000	0.10000

The flow rate is

$$\dot{m} = \rho AMc = \frac{P\sqrt{k}}{\sqrt{RT}}\frac{\pi \times D^2}{4}M = \frac{2000000\sqrt{1.4}}{\sqrt{300 \times 287}}\pi \times 0.2^2 \times 0.0419$$
$$\simeq 42.46[kg/sec]$$

End Solution

In this chapter, there are no examples on isothermal with supersonic flow.

9.8 Unchoked Situations in Fanno Flow

Table -9.2. The flow parameters for unchoked flow

$\mathbf{M_1}$	$\mathbf{M_2}$	$\frac{\mathbf{4fL}}{\mathbf{D}}\Big\vert_{\mathbf{max}}\Big\vert_{\mathbf{1}}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$
0.7272	0.84095	0.05005	0.05000	0.10000
0.6934	0.83997	0.08978	0.08971	0.10000
0.6684	0.84018	0.12949	0.12942	0.10000
0.6483	0.83920	0.16922	0.16912	0.10000
0.5914	0.83889	0.32807	0.32795	0.10000
0.5807	0.83827	0.36780	0.36766	0.10000
0.5708	0.83740	0.40754	0.40737	0.10000

9.8.1 Reynolds Number Effect

One of the interesting feature of the isothermal flow is that Reynolds number remains constant during the flow for an ideal gas material (enthalpy is a function of only the temperature). This fact simplifies the calculation of the friction factor. This topic has more discussion on the web than on "scientific" literature. Here is a theoretical example for such calculation that was discussed on the web.

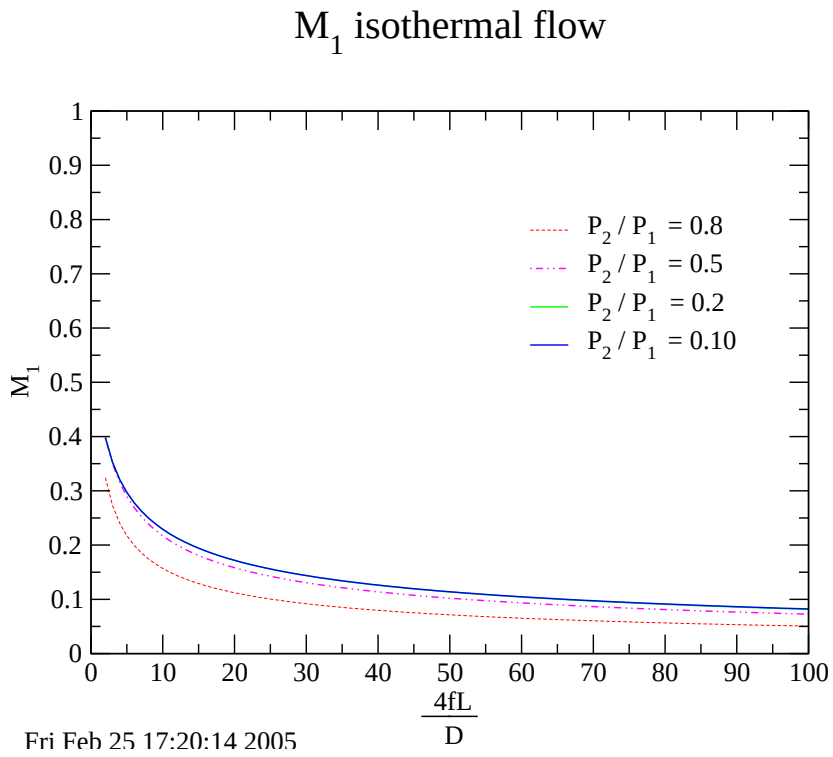


Fig. -9.3. The Mach number at the entrance to a tube under isothermal flow model as a function $\frac{4fL}{D}$.

Example 9.5:

Air flows in a tube with 0.1$[m]$ *diameter and 100*$[m]$ *in length. The relative roughness,* $\epsilon/D = 0.001$ *and the entrance pressure is* $P_1 = 20[Bar]$ *and the exit pressure is* $P_1 = 1[Bar]$ *. The surroundings temperature is* $27°\mathrm{C}$. *Estimate whether the flow is laminar or turbulent, estimate the friction factor, the entrance and exit Mach numbers and the flow rate.*

SOLUTION

The first complication is the know what is flow regimes. The process is to assume that the flow is turbulent (long pipe). In this case, for large Reynolds number the friction factor is about 0.005. Now the iterative procedure as following;

Calculate the $\frac{4fL}{D}$.

$$\frac{4fL}{D} = \frac{4 \times 0.005 \times 100}{0.1} = 20$$

The flow is choked for this value and the given pressure ratio. Thus,

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0^*}}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0^*}}$
0.17185	20.0000	4.9179	3.1460	4.9179	0.88017

For this iteration the viscosity of the air is taken from the Basics of Fluid Mechanics by this author and the Reynolds number can be calculated as

$$Re = \frac{D\,U\,\rho}{\mu} = \frac{0.1 \times 0.17185 \times \sqrt{1.4 \times 287 \times 300} \times \dfrac{200000}{287 \times 300}}{0.0008} \sim 17159.15$$

For this Reynolds number the fiction factor can be estimated by using the full Colebrook's equation

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/D_{\mathrm{h}}}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right) \tag{9.55}$$

or the approximated Haaland's equation

$$\frac{1}{\sqrt{f}} = -1.8\log_{10}\left[\left(\frac{\varepsilon/D}{3.7}\right)^{1.11} + \frac{6.9}{\mathrm{Re}}\right] \tag{9.56}$$

which provide $f = 0.0053$ and it is a reasonable answer in one iteration. Repeating the iteration results in

$$\tfrac{4fL}{D} = \frac{4 \times 0.0053 \times 100}{0.1} = 21.2$$

with

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0^*}}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{T_0}}{\mathbf{T_0^*}}$
0.16689	21.4000	5.0640	3.2357	5.0640	0.87987

And the "improved" Reynolds number is

$$Re = \frac{0.1 \times 0.16689 \times \sqrt{1.4 \times 287 \times 300} \times \dfrac{200000}{287 \times 300}}{0.0008} \sim 16669.6$$

And the friction number is .0054 which very good estimate compare with the assumption that this model was built on.

End Solution

CHAPTER 10

Fanno Flow

An adiabatic flow with friction is named after Ginno Fanno a Jewish engineer. This model is the second pipe flow model described here. The main restriction for this model is that heat transfer is negligible and can be ignored [1]. This model is applicable to flow processes which are very fast compared to heat transfer mechanisms with small Eckert number.

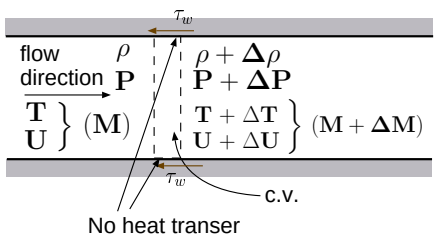


Fig. -10.1. Control volume of the gas flow in a constant cross section.

This model explains many industrial flow processes which includes emptying of pressured container through a relatively short tube, exhaust system of an internal combustion engine, compressed air systems, etc. As this model raised from the need to explain the steam flow in turbines.

10.1 Introduction

Consider a gas flowing through a conduit with a friction (see Figure (10.1)). It is advantages to examine the simplest situation and yet without losing the core properties of the process. Later, more general cases will be examined[2].

[1] Even the friction does not convert into heat

[2] Not ready yet, discussed on the ideal gas model and the entry length issues.

10.2 Fanno Model

The mass (continuity equation) balance can be written as

$$\begin{aligned} \dot{m} &= \rho\, A\, U = constant \\ \hookrightarrow \rho_1\, U_1 &= \rho_2\, U_2 \end{aligned} \tag{10.1}$$

The energy conservation (under the assumption that this model is adiabatic flow and the friction is not transformed into thermal energy) reads

$$\begin{aligned} T_{01} &= T_{02} \\ \hookrightarrow T_1 + \frac{{U_1}^2}{2\, c_p} &= T_2 + \frac{{U_2}^2}{2\, c_p} \end{aligned} \tag{10.2}$$

Or in a derivative from

$$C_p\, dT + d\left(\frac{U^2}{2}\right) = 0 \tag{10.3}$$

Again for simplicity, the perfect gas model is assumed[3].

$$\begin{aligned} P &= \rho R T \\ \hookrightarrow \frac{P_1}{\rho_1\, T_1} &= \frac{P_2}{\rho_2\, T_2} \end{aligned} \tag{10.4}$$

It is assumed that the flow can be approximated as one–dimensional. The force acting on the gas is the friction at the wall and the momentum conservation reads

$$-A\, dP - \tau_w dA_w = \dot{m} dU \tag{10.5}$$

It is convenient to define a hydraulic diameter as

$$D_H = \frac{4 \times \text{Cross Section Area}}{\text{wetted perimeter}} \tag{10.6}$$

Or in other words

$$A = \frac{\pi {D_H}^2}{4} \tag{10.7}$$

It is convenient to substitute D for D_H and yet it still will be referred to the same name as the hydraulic diameter. The infinitesimal area that shear stress is acting on is

$$dA_w = \pi D dx \tag{10.8}$$

[3]The equation of state is written again here so that all the relevant equations can be found when this chapter is printed separately.

Introducing the Fanning friction factor as a dimensionless friction factor which is some times referred to as the friction coefficient and reads as the following:

$$f = \frac{\tau_w}{\frac{1}{2}\,\rho\,U^2} \tag{10.9}$$

By utilizing equation (10.1) and substituting equation (10.9) into momentum equation (10.5) yields

$$-\frac{\overbrace{\pi\,D^2}^{A}}{4}\,dP - \pi\,D\,dx\,\overbrace{f\left(\frac{1}{2}\,\rho\,U^2\right)}^{\tau_w} = A\,\overbrace{\rho\,U}^{\frac{\dot{m}}{A}}\,dU \tag{10.10}$$

Dividing equation (10.10) by the cross section area, $\mathbf{A}$ and rearranging yields

$$-dP + \frac{4\,f\,dx}{D}\left(\frac{1}{2}\,\rho\,U^2\right) = \rho\,U\,dU \tag{10.11}$$

The second law is the last equation to be utilized to determine the flow direction.

$$s_2 \geq s_1 \tag{10.12}$$

10.3 Non–Dimensionalization of the Equations

Before solving the above equation a dimensionless process is applied. By utilizing the definition of the sound speed to produce the following identities for perfect gas

$$M^2 = \left(\frac{U}{c}\right)^2 = \frac{U^2}{k\,\underbrace{R\,T}_{\frac{P}{\rho}}} \tag{10.13}$$

Utilizing the definition of the perfect gas results in

$$M^2 = \frac{\rho\,U^2}{k\,P} \tag{10.14}$$

Using the identity in equation (10.13) and substituting it into equation (10.10) and after some rearrangement yields

$$-dP + \frac{4\,f\,dx}{D_H}\left(\frac{1}{2}\,k\,P\,M^2\right) = \frac{\rho U^2}{U}dU = \overbrace{k\,P\,M^2}^{\rho U^2}\,\frac{dU}{U} \tag{10.15}$$

By further rearranging equation (10.15) results in

$$-\frac{dP}{P} - \frac{4\,f\,dx}{D}\left(\frac{k\,M^2}{2}\right) = k\,M^2\frac{dU}{U} \tag{10.16}$$

It is convenient to relate expressions of (dP/P) and dU/U in terms of the Mach number and substituting it into equation (10.16). Derivative of mass conservation (10.1) results in

$$\frac{d\rho}{\rho} + \overbrace{\frac{1}{2}\frac{dU^2}{U^2}}^{\frac{dU}{U}} = 0 \tag{10.17}$$

Differentiating of the equation of state (10.4) and dividing the results by equation of state (10.4) results

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{dT} \tag{10.18}$$

Derivation of the Mach identity equation (10.13) and dividing by equation (10.13) yields

$$\frac{d(M^2)}{M^2} = \frac{d(U^2)}{U^2} - \frac{dT}{T} \tag{10.19}$$

Dividing the energy equation (10.3) by C_p and by utilizing the definition Mach number yields

$$\begin{aligned} &\frac{dT}{T} + \underbrace{\frac{1}{\left(\frac{k\,R}{(k-1)}\right)}}_{C_p}\frac{1}{T}\frac{U^2}{U^2}d\left(\frac{U^2}{2}\right) = \\ &\hookrightarrow \frac{dT}{T} + \frac{(k-1)}{\underbrace{k\,R\,T}_{c^2}}\frac{U^2}{U^2}\,d\left(\frac{U^2}{2}\right) = \\ &\hookrightarrow \frac{dT}{T} + \frac{k-1}{2}\,M^2\,\frac{dU^2}{U^2} = 0 \end{aligned} \tag{10.20}$$

Equations (10.16), (10.17), (10.18), (10.19), and (10.20) need to be solved. These equations are separable so one variable is a function of only single variable (the chosen as the independent variable). Explicit explanation is provided for only two variables, the rest variables can be done in a similar fashion. The dimensionless friction, $\frac{4\,f\,L}{D}$, is chosen as the independent variable since the change in the dimensionless resistance, $\frac{4\,f\,L}{D}$, causes the change in the other variables.

Combining equations (10.18) and (10.20) when eliminating dT/T results

$$\frac{dP}{P} = \frac{d\rho}{\rho} - \frac{(k-1)\,M^2}{2}\,\frac{dU^2}{U^2} \tag{10.21}$$

The term $\frac{d\rho}{\rho}$ can be eliminated by utilizing equation (10.17) and substituting it into equation (10.21) and rearrangement yields

$$\frac{dP}{P} = -\frac{1+(k-1)\,M^2}{2}\,\frac{dU^2}{U^2} \tag{10.22}$$

The term dU^2/U^2 can be eliminated by using (10.22)

$$\frac{dP}{P} = -\frac{k\,M^2\left(1+(k-1)M^2\right)}{2\,(1-M^2)}\frac{4\,f\,dx}{D} \tag{10.23}$$

The second equation for Mach number, M variable is obtained by combining equation (10.19) and (10.20) by eliminating dT/T. Then $d\rho/\rho$ and U are eliminated by utilizing equation (10.17) and equation (10.21). The only variable that is left is P (or dP/P) which can be eliminated by utilizing equation (10.23) and results in

$$\frac{4\,f\,dx}{D} = \frac{\left(1-M^2\right)\,dM^2}{k\,M^4(1+\dfrac{k-1}{2}\,M^2)} \tag{10.24}$$

Rearranging equation (10.24) results in

$$\frac{dM^2}{M^2} = \frac{k\,M^2\,\left(1+\dfrac{k-1}{2}\,M^2\right)}{1-M^2}\frac{4\,f\,dx}{D} \tag{10.25}$$

After similar mathematical manipulation one can get the relationship for the velocity to read

$$\frac{dU}{U} = \frac{k\,M^2}{2\,(1-M^2)}\frac{4\,f\,dx}{D} \tag{10.26}$$

and the relationship for the temperature is

$$\frac{dT}{T} = \frac{1}{2}\,\frac{dc}{c} = -\frac{k\,(k-1)\,M^4}{2\,(1-M^2)}\frac{4\,f\,dx}{D} \tag{10.27}$$

density is obtained by utilizing equations (10.26) and (10.17) to obtain

$$\frac{d\rho}{\rho} = -\frac{k\,M^2}{2\,(1-M^2)}\frac{4\,f\,dx}{D} \tag{10.28}$$

The stagnation pressure is similarly obtained as

$$\frac{dP_0}{P_0} = -\frac{k\,M^2}{2}\frac{4\,f\,dx}{D} \tag{10.29}$$

The second law reads

$$ds = C_p\,\ln\left(\frac{dT}{T}\right) - R\,\ln\left(\frac{dP}{P}\right) \tag{10.30}$$

The stagnation temperature expresses as $T_0 = T(1+(1-k)/2M^2)$. Taking derivative of this expression when M remains constant yields $dT_0 = dT(1+(1-k)/2M^2)$ and thus when these equations are divided they yield

$$dT/T = dT_0/T_0 \tag{10.31}$$

In similar fashion the relationship between the stagnation pressure and the pressure can be substituted into the entropy equation and result in

$$ds = C_p \ln\left(\frac{dT_0}{T_0}\right) - R \ln\left(\frac{dP_0}{P_0}\right) \tag{10.32}$$

The first law requires that the stagnation temperature remains constant, $(dT_0 = 0)$. Therefore the entropy change is

$$\frac{ds}{C_p} = -\frac{(k-1)}{k}\frac{dP_0}{P_0} \tag{10.33}$$

Using the equation for stagnation pressure the entropy equation yields

$$\frac{ds}{C_p} = \frac{(k-1)\,M^2}{2}\frac{4\,f\,dx}{D} \tag{10.34}$$

10.4 The Mechanics and Why the Flow is Choked?

The trends of the properties can be examined by looking in equations (10.23) through (10.33). For example, from equation (10.23) it can be observed that the critical point is when $M = 1$. When $M < 1$ the pressure decreases downstream as can be seen from equation (10.23) because fdx and M are positive. For the same reasons, in the supersonic branch, $M > 1$, the pressure increases downstream. This pressure increase is what makes compressible flow so different from "conventional" flow. Thus the discussion will be divided into two cases: One, flow above speed of sound. Two, flow with speed below the speed of sound.

Why the flow is choked?

Here, the explanation is based on the equations developed earlier and there is no known explanation that is based on the physics. First, it has to be recognized that the critical point is when $M = 1$. It will be shown that a change in location relative to this point change the trend and it is singular point by itself. For example, $dP(@M = 1) = \infty$ and mathematically it is a singular point (see equation (10.23)). Observing from equation (10.23) that increase or decrease from subsonic just below one $M = (1 - \epsilon)$ to above just above one $M = (1 + \epsilon)$ requires a change in a sign pressure direction. However, the pressure has to be a monotonic function which means that flow cannot crosses over the point of $M = 1$. This constrain means that because the flow cannot "crossover" $M = 1$ the gas has to reach to this speed, $M = 1$ at the last point. This situation is called choked flow.

The Trends

The trends or whether the variables are increasing or decreasing can be observed from looking at the equation developed. For example, the pressure can be examined by looking at equation (10.25). It demonstrates that the Mach number increases downstream

when the flow is subsonic. On the other hand, when the flow is supersonic, the pressure decreases.

The summary of the properties changes on the sides of the branch

	Subsonic	Supersonic
Pressure, P	decrease	increase
Mach number, M	increase	decrease
Velocity, U	increase	decrease
Temperature, T	decrease	increase
Density, ρ	decrease	increase

10.5 The Working Equations

Integration of equation (10.24) yields

Fanno FLD–M

$$\frac{4}{D}\int_L^{L_{max}} f dx = \frac{1}{k}\frac{1-M^2}{M^2} + \frac{k+1}{2k}\ln\frac{\frac{k+1}{2}M^2}{1+\frac{k-1}{2}M^2} \tag{10.35}$$

A representative friction factor is defined as

$$\bar{f} = \frac{1}{L_{max}}\int_0^{L_{max}} f dx \tag{10.36}$$

In the isothermal flow model it was shown that friction factor is constant through the process if the fluid is ideal gas. Here, the Reynolds number defined in equation (9.28) is not constant because the temperature is not constant. The viscosity even for ideal gas is complex function of the temperature (further reading in "Basic of Fluid Mechanics" chapter one, Potto Project). However, the temperature variation is very limited. Simple improvement can be done by assuming constant constant viscosity (constant friction factor) and find the temperature on the two sides of the tube to improve the friction factor for the next iteration. The maximum error can be estimated by looking at the maximum change of the temperature. The temperature can be reduced by less than 20% for most range of the specific heats ratio. The viscosity change for this change is for many gases about 10%. For these gases the maximum increase of average Reynolds number is only 5%. What this change in Reynolds number does to friction factor? That depend in the range of Reynolds number. For Reynolds number larger than 10,000 the change in friction factor can be considered negligible. For the other extreme, laminar flow it can estimated that change of 5% in Reynolds number change about the same amount in friction factor. With the exception of the jump from a laminar flow to

a turbulent flow, the change is noticeable but very small. In the light of the about discussion the friction factor is assumed to constant. By utilizing the mean average theorem equation (10.35) yields

Resistance Mach Relationship

$$\frac{4\,\overline{f}\,L_{max}}{D} = \frac{1}{k}\left(\frac{1-M^2}{M^2}\right) + \frac{k+1}{2k}\,\ln\left(\frac{\frac{k+1}{2}\,M^2}{1+\frac{k-1}{2}\,M^2}\right) \tag{10.37}$$

It is common to replace the $\bar{f}$ with f which is adopted in this book.

Advance material can be skipped

For a very long pipe the value of the $\frac{4\,f\,L}{D}$ is large but the value the Mach number is very small. Hence equation (10.37) can be simplified for that case as following. The term change as $(1-M^2)/M^2 \sim 1/M^2$ and term in the parentheses after the $\ln$ in the parentheses is $\frac{k+1}{2}\,M^2$. Thus equation can be written for small Mach number as

$$\frac{4\,\overline{f}\,L_{max}}{D} = \frac{1}{k}\,\frac{1}{M^2} + \frac{k+1}{2\,k}\,\ln\left(\frac{k+1}{2}\,M^2\right) \tag{10.38}$$

Taylor series centered at 1 is

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}\cdots$$

This series does not converge very quickly and a mathematical trick is used where defining new variable η as

$$\eta = \frac{(x-1)}{(x+1)} \qquad \text{or} \qquad \eta = \frac{(M^2-1)}{(M^2+1)}$$

Thus returning to original equation yields

$$\ln(x) = \ln\left(\frac{1+\eta}{1-\eta}\right)$$

The right hand side can be expanded as

$$\ln\left(\frac{1+\eta}{1-\eta}\right) = 2\,\eta\,\left(\frac{1}{1} + \frac{1}{3}\eta^2 + \frac{1}{5}\eta^4 + \frac{1}{7}\eta^6 + \frac{1}{9}\eta^8 + \cdots\right)$$

For small M^2 (therefor for η) the resistance can be approximated as

$$\frac{4\,\overline{f}\,L_{max}}{D} = \frac{1}{k}\,\frac{1}{M^2} + \frac{(k+1)\,\eta}{k}\,\left(\frac{1}{1} + \frac{1}{3}\eta^2 + \frac{1}{5}\eta^4 + \frac{1}{7}\eta^6 + \frac{1}{9}\eta^8 + \cdots\right) + \frac{(k+1)}{k}\,\ln\frac{k+1}{2} \tag{10.39}$$

End Advance material

Equations (10.23), (10.26), (10.27), (10.28), (10.28), and (10.29) can be solved. For example, the pressure as written in equation (10.22) is represented by $\frac{4fL}{D}$, and Mach number. Now equation (10.23) can eliminate term $\frac{4fL}{D}$ and describe the pressure on the Mach number. Dividing equation (10.23) in equation (10.25) yields

$$\frac{\frac{dP}{P}}{\frac{dM^2}{M^2}} = -\frac{1+(k-1M^2}{2\,M^2\left(1+\frac{k-1}{2}M^2\right)}\,dM^2 \tag{10.40}$$

The symbol "*" denotes the state when the flow is choked and Mach number is equal to 1. Thus, $M = 1$ when $P = P^*$ equation (10.40) can be integrated to yield:

Mach–Pressure Ratio

$$\frac{P}{P^*} = \frac{1}{M}\sqrt{\frac{\frac{k+1}{2}}{1+\frac{k-1}{2}M^2}} \tag{10.41}$$

In the same fashion the variables ratios can be obtained

Temperature Ratio

$$\frac{T}{T^*} = \frac{c^2}{c^{*2}} = \frac{\frac{k+1}{2}}{1+\frac{k-1}{2}M^2} \tag{10.42}$$

The density ratio is

Density Ratio

$$\frac{\rho}{\rho^*} = \frac{1}{M}\sqrt{\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}} \tag{10.43}$$

The velocity ratio is

Velocity Ratio

$$\frac{U}{U^*} = \left(\frac{\rho}{\rho^*}\right)^{-1} = M\sqrt{\frac{\frac{k+1}{2}}{1+\frac{k-1}{2}M^2}} \tag{10.44}$$

The stagnation pressure decreases and can be expressed by

$$\frac{P_0}{{P_0}^*} = \frac{\overbrace{\frac{P_0}{P}}^{\left(1+\frac{1-k}{2}M^2\right)^{\frac{k}{k-1}}}}{\underbrace{\frac{{P_0}^*}{P^*}}_{\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}}} \frac{P}{P^*} \tag{10.45}$$

Using the pressure ratio in equation (10.41) and substituting it into equation (10.45) yields

$$\frac{P_0}{{P_0}^*} = \left(\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k}{k-1}} \frac{1}{M}\sqrt{\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}} \tag{10.46}$$

And further rearranging equation (10.46) provides

Stagnation Pressure Ratio

$$\frac{P_0}{{P_0}^*} = \frac{1}{M}\left(\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right)^{\frac{k+1}{2(k-1)}} \tag{10.47}$$

The integration of equation (10.33) yields

$$\frac{s-s^*}{C_p} = \ln M^2 \sqrt{\left(\frac{k+1}{2M^2\left(1+\frac{k-1}{2}M^2\right)}\right)^{\frac{k+1}{k}}} \tag{10.48}$$

The results of these equations are plotted in Figure 10.2

The Fanno flow is in many cases shockless and therefore a relationship between two points should be derived. In most times, the "star" values are imaginary values that represent the value at choking. The real ratio can be obtained by two star ratios as an example

$$\frac{T_2}{T_1} = \frac{\left.\frac{T}{T^*}\right|_{M_2}}{\left.\frac{T}{T^*}\right|_{M_1}} \tag{10.49}$$

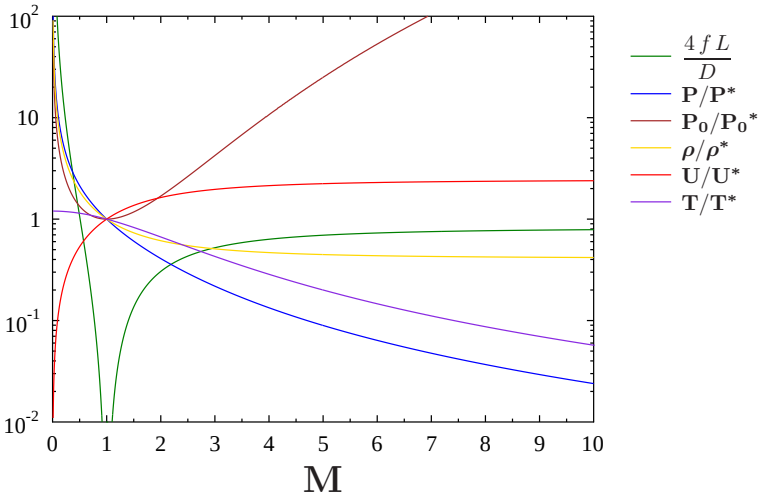


Fig. -10.2. Various parameters in Fanno flow shown as a function of Mach number.

A special interest is the equation for the dimensionless friction as following

$$\int_{L_1}^{L_2} \frac{4\,f\,L}{D}dx = \int_{L_1}^{L_{max}} \frac{4\,f\,L}{D}dx - \int_{L_2}^{L_{max}} \frac{4\,f\,L}{D}dx \tag{10.50}$$

Hence,

fld Working Equation

$$\left(\frac{4\,f\,L_{max}}{D}\right)_2 = \left(\frac{4\,f\,L_{max}}{D}\right)_1 - \frac{4\,f\,L}{D} \tag{10.51}$$

10.6 Examples of Fanno Flow

Example 10.1:

Air flows from a reservoir and enters a uniform pipe with a diameter of 0.05 [m] and length of 10 [m]. The air exits to the atmosphere. The following conditions prevail at the exit: $P_2 = 1[bar]$ temperature $T_2 = 27°C$ $M_2 = 0.9$[4]. Assume that the average friction factor to be $f = 0.004$ and that the flow from the reservoir up to the pipe inlet is essentially isentropic. Estimate the total temperature and total pressure in the reservoir under the Fanno flow model.

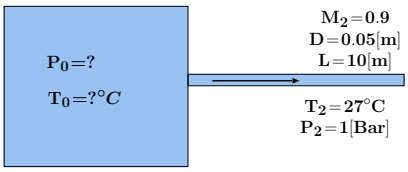


Fig. -10.3. Schematic of Example 10.1.

SOLUTION

For isentropic, the flow to the pipe inlet, the temperature and the total pressure at the pipe inlet are the same as those in the reservoir. Thus, finding the star pressure and temperature at the pipe inlet is the solution. With the Mach number and temperature known at the exit, the total temperature at the entrance can be obtained by knowing the $\frac{4fL}{D}$. For given Mach number ($M = 0.9$) the following is obtained.

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.90000	0.01451	1.1291	1.0089	1.0934	0.9146	1.0327

So, the total temperature at the exit is

$$T^*|_2 = \left.\frac{T^*}{T}\right|_2 T_2 = \frac{300}{1.0327} = 290.5[K]$$

To "move" to the other side of the tube the $\frac{4fL}{D}$ is added as

$$\left.\frac{4fL}{D}\right|_1 = \frac{4fL}{D} + \left.\frac{4fL}{D}\right|_2 = \frac{4 \times 0.004 \times 10}{0.05} + 0.01451 \simeq 3.21$$

The rest of the parameters can be obtained with the new $\frac{4fL}{D}$ either from Table (10.1) by interpolations or by utilizing the attached program.

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.35886	3.2100	3.0140	1.7405	2.5764	0.38814	1.1699

Note that the subsonic branch is chosen. The stagnation ratios has to be added for $M = 0.35886$

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.35886	0.97489	0.93840	1.7405	0.91484	1.5922	0.78305

The total pressure P_{01} can be found from the combination of the ratios as follows:

$$\begin{aligned} P_{01} =& \overbrace{P_2 \left.\frac{P^*}{P}\right|_2 \left.\frac{P}{P^*}\right|_1}^{P^*} \overbrace{\left.\frac{P_0}{P}\right|_1}^{P_1} \\ =& 1 \times \frac{1}{1.12913} \times 3.014 \times \frac{1}{0.915} = 2.91[Bar] \end{aligned}$$

[4] This property is given only for academic purposes. There is no Mach meter.

$$T_{01} = \overbrace{T_2 \left.\frac{T^*}{T}\right|_2}^{T_1} \overbrace{\left.\frac{T}{T^*}\right|_1}^{T^*} \left.\frac{T_0}{T}\right|_1$$
$$= 300 \times \frac{1}{1.0327} \times 1.17 \times \frac{1}{0.975} \simeq 348K = 75^\circ\mathrm{C}$$

End Solution

Another academic question/example:

Example 10.2:

A system is composed of a convergent-divergent nozzle followed by a tube with length of 2.5 [cm] in diameter and 1.0 [m] long. The system is supplied by a vessel. The vessel conditions are at 29.65 [Bar], 400 K. With these conditions a pipe inlet Mach number is 3.0. A normal shock wave occurs in the tube and the flow discharges to the atmosphere, determine:

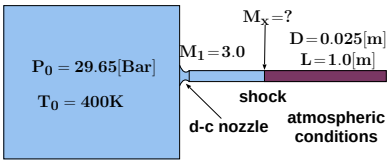


Fig. -10.4. The schematic of Example (10.2).

(a) the mass flow rate through the system;

(b) the temperature at the pipe exit; and

(c) determine the Mach number when a normal shock wave occurs [M_x].

Take k = 1.4, R = 287 $[J/kgK]$ and $f = 0.005$.

SOLUTION

(a) Assuming that the pressure vessel is very much larger than the pipe, therefore the velocity in the vessel can be assumed to be small enough so it can be neglected. Thus, the stagnation conditions can be approximated for the condition in the tank. It is further assumed that the flow through the nozzle can be approximated as isentropic. Hence, $T_{01} = 400K$ and $P_{01} = 29.65[Par]$.

The mass flow rate through the system is constant and for simplicity point 1 is chosen in which,

$$\dot{m} = \rho\, A\, M\, c$$

The density and speed of sound are unknowns and need to be computed. With the isentropic relationship, the Mach number at point one (1) is known, then the following can be found either from Table 10.1, or the popular Potto–GDC as

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
3.0000	0.35714	0.07623	4.2346	0.02722	0.11528	0.65326

The temperature is

$$T_1 = \frac{T_1}{T_{01}} T_{01} = 0.357 \times 400 = 142.8K$$

Using the temperature, the speed of sound can be calculated as

$$c_1 = \sqrt{k\, R\, T} = \sqrt{1.4 \times 287 \times 142.8} \simeq 239.54[m/sec]$$

The pressure at point 1 can be calculated as

$$P_1 = \frac{P_1}{P_{01}} P_{01} = 0.027 \times 30 \simeq 0.81[Bar]$$

The density as a function of other properties at point 1 is

$$\rho_1 = \left.\frac{P}{RT}\right|_1 = \frac{8.1 \times 10^4}{287 \times 142.8} \simeq 1.97 \left[\frac{kg}{m^3}\right]$$

The mass flow rate can be evaluated from equation (10.1)

$$\dot{m} = 1.97 \times \frac{\pi \times 0.025^2}{4} \times 3 \times 239.54 = 0.69 \left[\frac{kg}{sec}\right]$$

(b) First, check whether the flow is shockless by comparing the flow resistance and the maximum possible resistance. From the Table 10.1 or by using the famous Potto–GDC, is to obtain the following

$\mathbf{M}$	$\frac{\mathbf{4\,f\,L}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
3.0000	0.52216	0.21822	4.2346	0.50918	1.9640	0.42857

and the conditions of the tube are

$$\frac{4fL}{D} = \frac{4 \times 0.005 \times 1.0}{0.025} = 0.8$$

Since $0.8 > 0.52216$ the flow is choked and with a shock wave.

The exit pressure determines the location of the shock, if a shock exists, by comparing "possible" P_{exit} to P_B. Two possibilities are needed to be checked; one, the shock at the entrance of the tube, and two, shock at the exit and comparing the pressure ratios. First, the possibility that the shock wave occurs immediately at the entrance for which the ratio for M_x are (shock wave Table 6.2)

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.0000	0.47519	2.6790	3.8571	10.3333	0.32834

After the shock wave the flow is subsonic with "M_1"$= 0.47519$. (Fanno flow Table 10.1)

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0{}^*}}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.47519	1.2919	2.2549	1.3904	1.9640	0.50917	1.1481

The stagnation values for $M = 0.47519$ are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.47519	0.95679	0.89545	1.3904	0.85676	1.1912	0.65326

The ratio of exit pressure to the chamber total pressure is

$$\begin{aligned}
\frac{P_2}{P_0} &= \overbrace{\left(\frac{P_2}{P^*}\right)}^{1}\left(\frac{P^*}{P_1}\right)\left(\frac{P_1}{P_{0_y}}\right)\left(\frac{P_{0_y}}{P_{0_x}}\right)\overbrace{\left(\frac{P_{0_x}}{P_0}\right)}^{1} \\
&= 1 \times \frac{1}{2.2549} \times 0.8568 \times 0.32834 \times 1 \\
&= 0.12476
\end{aligned}$$

The actual pressure ratio $1/29.65 = 0.0338$ is smaller than the case in which shock occurs at the entrance. Thus, the shock is somewhere downstream. One possible way to find the exit temperature, T_2 is by finding the location of the shock. To find the location of the shock ratio of the pressure ratio, $\frac{P_2}{P_1}$ is needed. With the location of shock, "claiming" upstream from the exit through shock to the entrance. For example, calculate the parameters for shock location with known $\frac{4fL}{D}$ in the "y" side. Then either by utilizing shock table or the program, to obtain the upstream Mach number.

The procedure for the calculations:

Calculate the entrance Mach number assuming the shock occurs at the exit:

1) a) set $M_2^{'} = 1$ assume the flow in the entire tube is supersonic:
   b) calculated $M_1^{'}$

   Note this Mach number is the high Value.

   Calculate the entrance Mach assuming shock at the entrance.

2) a) set $M_2 = 1$
   b) add $\frac{4fL}{D}$ and calculated M_1' for subsonic branch
   c) calculated M_x for M_1'

   Note this Mach number is the low Value.

   According your root finding algorithm[5] calculate or guess the shock location and then compute as above the new M_1.

3) a) set $M_2 = 1$
   b) for the new $\frac{4fL}{D}$ and compute the new M_y' for the subsonic branch
   c) calculated M_x' for the M_y'
   d) Add the leftover of $\frac{4fL}{D}$ and calculated the M_1

4) guess new location for the shock according to your finding root procedure and according to the result, repeat previous stage until the solution is obtained.

$\mathbf{M_1}$	$\mathbf{M_2}$	$\frac{\mathbf{4fL}}{\mathbf{D}}\big\vert_{\mathbf{up}}$	$\frac{\mathbf{4fL}}{\mathbf{D}}\big\vert_{\mathbf{down}}$	$\mathbf{M_x}$	$\mathbf{M_y}$
3.0000	1.0000	0.22019	0.57981	1.9899	0.57910

(c) The way of the numerical procedure for solving this problem is by finding $\frac{4fL}{D}\Big\vert_{up}$ that will produce $M_1 = 3$. In the process M_x and M_y must be calculated (see the chapter on the program with its algorithms.).

End Solution

[5] You can use any method you which, but be-careful second order methods like Newton-Rapson method can be unstable.

10.7 Supersonic Branch

In Chapter (9) it was shown that the isothermal model cannot describe adequately the situation because the thermal entry length is relatively large compared to the pipe length and the heat transfer is not sufficient to maintain constant temperature. In the Fanno model there is no heat transfer, and, furthermore, because the very limited amount of heat transformed it is closer to an adiabatic flow. The only limitation of the model is its uniform velocity (assuming parabolic flow for laminar and different profile for turbulent flow.). The information from the wall to the tube center[6] is slower in reality. However, experiments from many starting with 1938 work by Frossel[7] has shown that the error is not significant. Nevertheless, the comparison with reality shows that heat transfer cause changes to the flow and they need/should to be expected. These changes include the choking point at lower Mach number.

10.8 Maximum Length for the Supersonic Flow

It has to be noted and recognized that as opposed to subsonic branch the supersonic branch has a limited length. It also must be recognized that there is a maximum length for which only supersonic flow can exist[8]. These results were obtained from the mathematical derivations but were verified by numerous experiments[9]. The maximum length of the supersonic can be evaluated when $M = \infty$ as follows:

$$\begin{aligned}
\frac{4\,f\,L_{max}}{D} &= \frac{1-M^2}{kM^2} + \frac{k+1}{2k}\ln\frac{\frac{k+1}{2}M^2}{2\left(1+\frac{k-1}{2}M^2\right)} = \\
\tfrac{4fL}{D}\,(M\to\infty) &\sim \frac{-\infty}{k\times\infty} + \frac{k+1}{2k}\ln\frac{(k+1)\infty}{(k-1)\infty} \\
&= \frac{-1}{k} + \frac{k+1}{2\,k}\ln\frac{(k+1)}{(k-1)} \\
&= \tfrac{4fL}{D}(M\to\infty, k=1.4) = 0.8215
\end{aligned}$$

$$\frac{4\,f\,L_{max}}{D} = \tfrac{4fL}{D}(M\to\infty, k=1.4) = 0.8215 \tag{10.52}$$

The maximum length of the supersonic flow is limited by the above number. From the above analysis, it can be observed that no matter how high the entrance Mach number will be the tube length is limited and depends only on specific heat ratio, k as shown in Figure (10.5).

[6] The word information referred to is the shear stress transformed from the wall to the center of the tube.

[7] See on the web `http://naca.larc.nasa.gov/digidoc/report/tm/44/NACA-TM-844.PDF`

[8] Many in the industry have difficulties in understanding this concept. The author seeks for a nice explanation of this concept for non–fluid mechanics engineers. This solicitation is about how to explain this issue to non-engineers or engineer without a proper background.

[9] If you have experiments demonstrating this point, please provide to the undersign so they can be added to this book. Many of the pictures in the literature carry copyright statements and thus can be presented here.

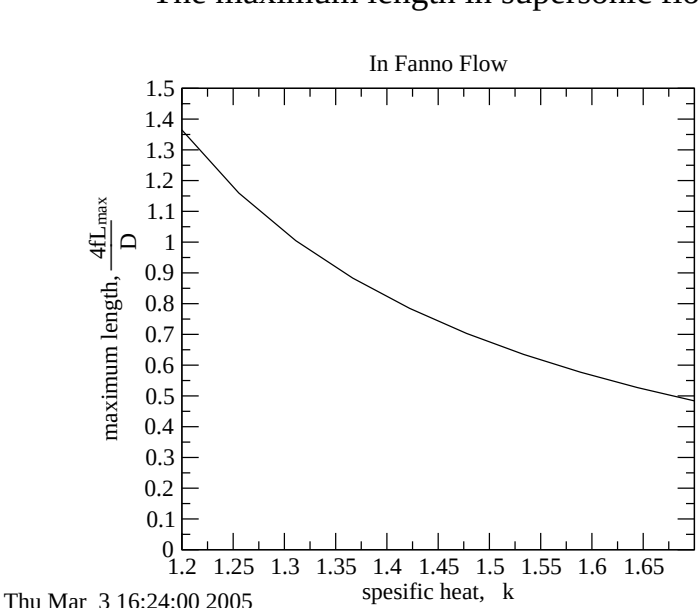


Fig. -10.5. The maximum length as a function of specific heat, k.

10.9 Working Conditions

It has to be recognized that there are two regimes that can occur in Fanno flow model one of subsonic flow and the other supersonic flow. Even the flow in the tube starts as a supersonic in parts of the tube can be transformed into the subsonic branch. A shock wave can occur and some portions of the tube will be in a subsonic flow pattern.

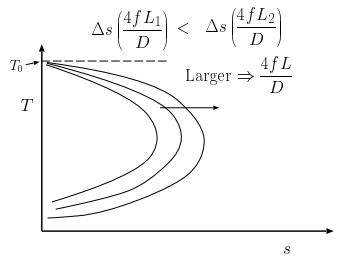


Fig. -10.6. The effects of increase of $\frac{4fL}{D}$ on the Fanno line.

The discussion has to differentiate between two ways of feeding the tube: converging nozzle or a converging-diverging nozzle. Three parameters, the dimensionless friction, $\frac{4fL}{D}$, the entrance Mach number, M_1, and the pressure ratio, P_2/P_1 are controlling the flow. Only a combination of these two parameters is truly independent. However, all the three parameters can be varied and they are discussed separately here.

10.9.1 Variations of The Tube Length ($\frac{4fL}{D}$) Effects

In the analysis of this effect, it should be assumed that back pressure is constant and/or low as possible as needed to maintain a choked flow. First, the treatment of the two branches are separated.

Fanno Flow Subsonic branch

For converging nozzle feeding, increasing the tube length results in increasing the exit Mach number (normally denoted herein as M_2). Once the Mach number reaches maximum ($M = 1$), no further increase of the exit Mach number can be achieved. In this process, the mass flow rate decreases.

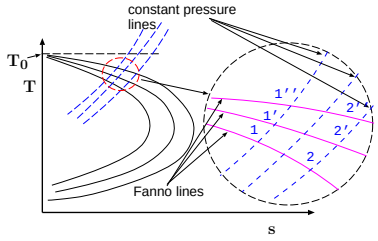


Fig. -10.7. The effects of the increase of $\frac{4fL}{D}$ on the Fanno Line.

It is worth noting that entrance Mach number is reduced (as some might explain it to reduce the flow rate). The entrance temperature increases as can be seen from Figure (10.7). The velocity therefore must decrease because the loss of the enthalpy (stagnation temperature) is "used." The density decrease because $\rho = \frac{P}{RT}$ and when pressure is remains almost constant the density decreases. Thus, the mass flow rate must decrease. These results are applicable to the converging nozzle.

In the case of the converging–diverging feeding nozzle, increase of the dimensionless friction, $\frac{4fL}{D}$, results in a similar flow pattern as in the converging nozzle. Once the flow becomes choked a different flow pattern emerges.

Fanno Flow Supersonic Branch

There are several transitional points that change the pattern of the flow. Point **a** is the choking point (for the supersonic branch) in which the exit Mach number reaches to one. Point **b** is the maximum possible flow for supersonic flow and is not dependent on the nozzle. The next point, referred here as the critical point **c**, is the point in which no supersonic flow is possible in the tube i.e. the shock reaches to the nozzle. There is another point **d**, in which no supersonic flow is possible in the entire nozzle–tube system. Between these transitional points the effect parameters such as mass flow rate, entrance and exit Mach number are discussed.

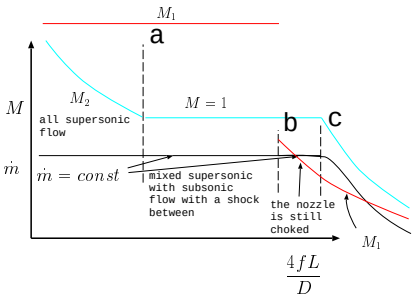


Fig. -10.8. The Mach numbers at entrance and exit of tube and mass flow rate for Fanno Flow as a function of the $\frac{4fL}{D}$.

At the starting point the flow is choked in the nozzle, to achieve supersonic flow. The following ranges that has to be discussed includes (see Figure (10.8)):

$$\begin{array}{rcccll}
0 & < & \frac{4fL}{D} & < & \left(\frac{4fL}{D}\right)_{choking} & 0 \to \mathbf{a} \\
\left(\frac{4fL}{D}\right)_{choking} & < & \frac{4fL}{D} & < & \left(\frac{4fL}{D}\right)_{shockless} & \mathbf{a} \to \mathbf{b} \\
\left(\frac{4fL}{D}\right)_{shockless} & < & \frac{4fL}{D} & < & \left(\frac{4fL}{D}\right)_{chokeless} & \mathbf{b} \to \mathbf{c} \\
\left(\frac{4fL}{D}\right)_{chokeless} & < & \frac{4fL}{D} & < & \infty & \mathbf{c} \to \infty
\end{array}$$

The 0-$\mathbf{a}$ range, the mass flow rate is constant because the flow is choked at the nozzle. The entrance Mach number, M_1 is constant because it is a function of the nozzle design only. The exit Mach number, M_2 decreases (remember this flow is on the supersonic branch) and starts ($\frac{4fL}{D} = 0$) as $M_2 = M_1$. At the end of the range $\mathbf{a}$, $M_2 = 1$. In the range of $\mathbf{a} - \mathbf{b}$ the flow is all supersonic.

In the next range $\mathbf{a} - \mathbf{b}$ The flow is double choked and make the adjustment for the flow rate at different choking points by changing the shock location. The mass flow rate continues to be constant. The entrance Mach continues to be constant and exit Mach number is constant.

The total maximum available for supersonic flow $\mathbf{b} - \mathbf{b}'$, $\left(\frac{4fL}{D}\right)_{max}$, is only a theoretical length in which the supersonic flow can occur if nozzle is provided with a larger Mach number (a change to the nozzle area ratio which also reduces the mass flow rate). In the range $\mathbf{b} - \mathbf{c}$, it is a more practical point.

In semi supersonic flow $\mathbf{b} - \mathbf{c}$ (in which no supersonic is available in the tube but only in the nozzle) the flow is still double choked and the mass flow rate is constant. Notice that exit Mach number, M_2 is still one. However, the entrance Mach number, M_1, reduces with the increase of $\frac{4fL}{D}$.

It is worth noticing that in the $\mathbf{a} - \mathbf{c}$ the mass flow rate nozzle entrance velocity and the exit velocity remains constant![10]

In the last range $\mathbf{c} - \infty$ the end is really the pressure limit or the break of the model and the isothermal model is more appropriate to describe the flow. In this range, the flow rate decreases since ($\dot{m} \propto M_1$)[11].

To summarize the above discussion, Figures (10.8) exhibits the development of M_1, M_2 mass flow rate as a function of $\frac{4fL}{D}$. Somewhat different then the subsonic branch the mass flow rate is constant even if the flow in the tube is completely subsonic. This situation is because of the "double" choked condition in the nozzle. The exit Mach

[10] On a personal note, this situation is rather strange to explain. On one hand, the resistance increases and on the other hand, the exit Mach number remains constant and equal to one. Does anyone have an explanation for this strange behavior suitable for non–engineers or engineers without background in fluid mechanics?

[11] Note that ρ_1 increases with decreases of M_1 but this effect is less significant.

M_2 is a continuous monotonic function that decreases with $\frac{4fL}{D}$. The entrance Mach M_1 is a non continuous function with a jump at the point when shock occurs at the entrance "moves" into the nozzle.

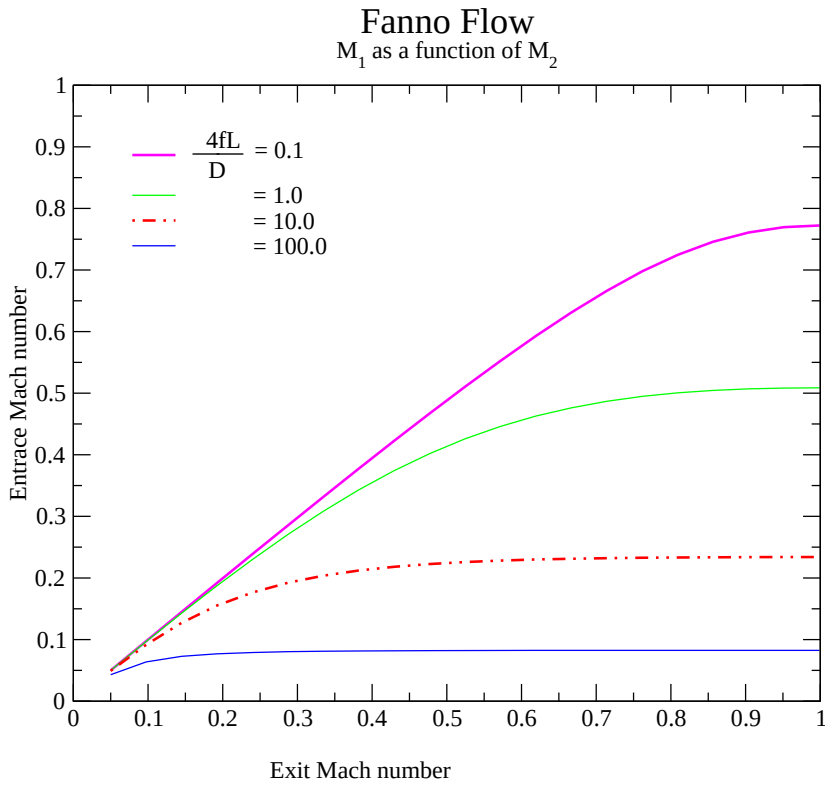


Fig. -10.9. M_1 as a function M_2 for various $\frac{4fL}{D}$.

Figure 10.9 exhibits the M_1 as a function of M_2. The Figure was calculated by utilizing the data from Figure (10.2) by obtaining the $\frac{4fL}{D}\Big|_{max}$ for M_2 and subtracting the given $\frac{4fL}{D}$ and finding the corresponding M_1.

The Figure (10.10) exhibits the entrance Mach number as a function of the M_2. Obviously there can be two extreme possibilities for the subsonic exit branch. Subsonic velocity occurs for supersonic entrance velocity, one, when the shock wave occurs at the tube exit and two, at the tube entrance. In Figure (10.10) only for $\frac{4fL}{D} = 0.1$ and $\frac{4fL}{D} = 0.4$ two extremes are shown. For $\frac{4fL}{D} = 0.2$ shown with only shock at the exit only. Obviously, and as can be observed, the larger $\frac{4fL}{D}$ creates larger differences between exit Mach number for the different shock locations. The larger $\frac{4fL}{D}$ larger M_1

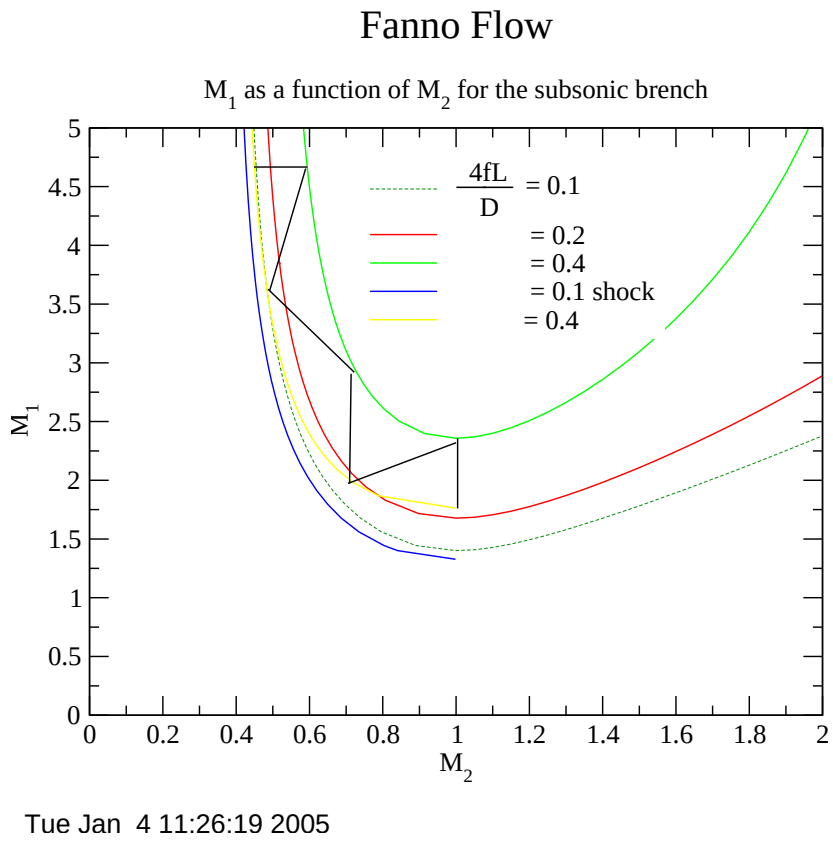


Fig. -10.10. M_1 *as a function* M_2 *for different* $\frac{4fL}{D}$ *for supersonic entrance velocity.*

must occurs even for shock at the entrance.

For a given $\frac{4\,f\,L}{D}$, below the maximum critical length, the supersonic entrance flow has three different regimes which depends on the back pressure. One, shockless flow, tow, shock at the entrance, and three, shock at the exit. Below, the maximum critical length is mathematically

$$\frac{4\,f\,L}{D} > -\frac{1}{k} + \frac{1+k}{2\,k} \ln\left(\frac{k+1}{k-1}\right)$$

For cases of $\frac{4\,f\,L}{D}$ above the maximum critical length no supersonic flow can be over the whole tube and at some point a shock will occur and the flow becomes subsonic flow[12].

[12] See more on the discussion about changing the length of the tube.

10.9.2 The Pressure Ratio, P_2/P_1, effects

In this section the studied parameter is the variation of the back pressure and thus, the pressure ratio (P_2/P_1) variations. For very low pressure ratio the flow can be assumed as incompressible with exit Mach number smaller than < 0.3. As the pressure ratio increases (smaller back pressure, P_2), the exit and entrance Mach numbers increase. According to Fanno model the value of $\frac{4fL}{D}$ is constant (friction factor, f, is independent of the parameters such as, Mach number, Reynolds number et cetera) thus the flow remains on the same Fanno line. For cases where the supply come from a reservoir with a constant pressure, the entrance pressure decreases as well because of the increase in the entrance Mach number (velocity).

Again a differentiation of the feeding is important to point out. If the feeding nozzle is converging than the flow will be only subsonic. If the nozzle is "converging–diverging" than in some part supersonic flow is possible. At first the converging nozzle is presented and later the converging-diverging nozzle is explained.

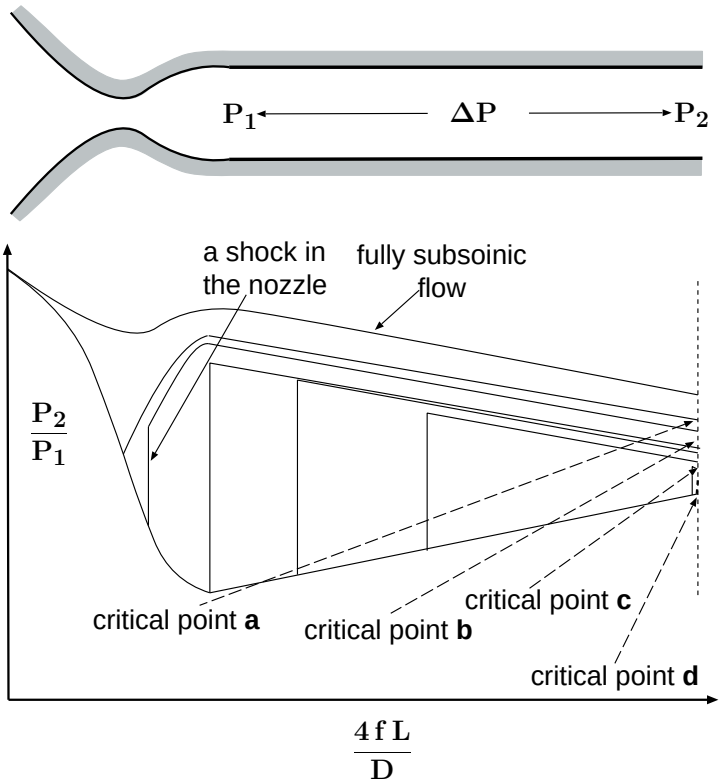


Fig. -10.11. The pressure distribution as a function of $\frac{4fL}{D}$ for a short $\frac{4fL}{D}$.

Choking explanation for pressure variation/reduction

Decreasing the pressure ratio or in actuality the back pressure, results in increase of the entrance and the exit velocity until a maximum is reached for the exit velocity. The maximum velocity is when exit Mach number equals one. The Mach number, as it was shown in Chapter (5), can increases only if the area increase. In our model the tube area is postulated as a constant therefore the velocity cannot increase any further. However, for the flow to be continuous the pressure must decrease and for that the velocity must increase. Something must break since there are conflicting demands and it result in a "jump" in the flow. This jump is referred to as a choked flow. Any additional reduction in the back pressure will not change the situation in the tube. The only change will be at tube surroundings which are irrelevant to this discussion.

If the feeding nozzle is a "converging–diverging" then it has to be differentiated between two cases; One case is where the $\frac{4fL}{D}$ is short or equal to the critical length. The critical length is the maximum $\left.\frac{4fL}{D}\right|_{max}$ that associate with entrance Mach number.

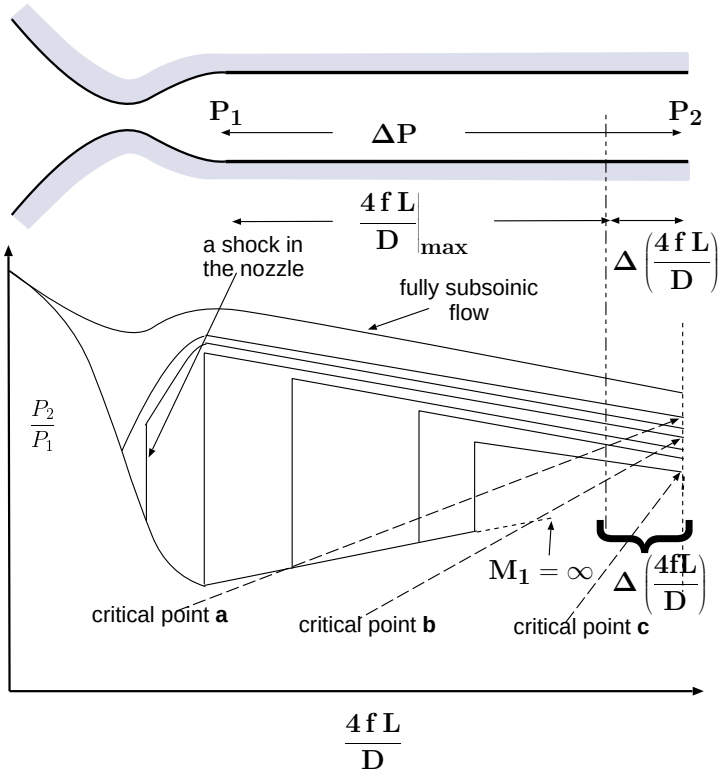


Fig. -10.12. The pressure distribution as a function of $\frac{4fL}{D}$ for a long $\frac{4fL}{D}$.

Short $4\,f\,L/\,D$

Figure 10.12 shows different pressure profiles for different back pressures. Before the flow reaches critical point **a** (in the Figure 10.12) the flow is subsonic. Up to this stage the nozzle feeding the tube increases the mass flow rate (with decreasing back pressure). Pressure between point **a** and point **b** the shock is in the nozzle. In this range and further reduction of the pressure the mass flow rate is constant no matter how low the back pressure is reduced. Once the back pressure is less than point **b** the supersonic reaches to the tube. Note however that exit Mach number, $M_2 < 1$ and is **not** 1. A back pressure that is at the critical point **c** results in a shock wave that is at the exit. When the back pressure is below point **c**, the tube is "clean" of any shock[13]. The back pressure below point **c** has some adjustment as it occurs with exceptions of point **d**.

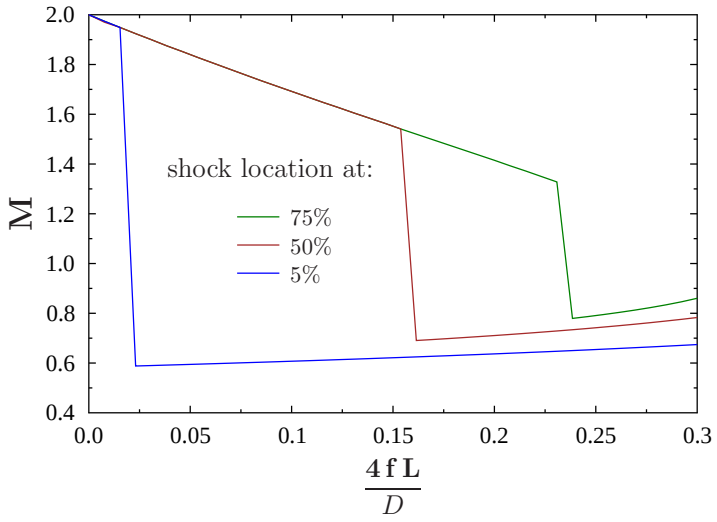


Fig. -10.13. The effects of pressure variations on Mach number profile as a function of $\frac{4\,f\,L}{D}$ *when the total resistance* $\frac{4\,f\,L}{D} = 0.3$ *for Fanno Flow.*

Long $\frac{4\,f\,L}{D}$

In the case of $\frac{4\,f\,L}{D} > \left.\frac{4\,f\,L}{D}\right|_{max}$ reduction of the back pressure results in the same process as explained in the short $\frac{4\,f\,L}{D}$ up to point **c**. However, point **c** in this case is different from point **c** at the case of short tube $\frac{4\,f\,L}{D} < \left.\frac{4\,f\,L}{D}\right|_{max}$. In this point the exit Mach number is equal to 1 and the flow is double shock. Further reduction of the

[13] It is common misconception that the back pressure has to be at point **d**.

back pressure at this stage will not "move" the shock wave downstream the nozzle. At point **c** or location of the shock wave, is a function entrance Mach number, M_1 and the "extra" $\frac{4fL}{D}$. There is no analytical solution for the location of this point **c**. The procedure is (will be) presented in later stage.

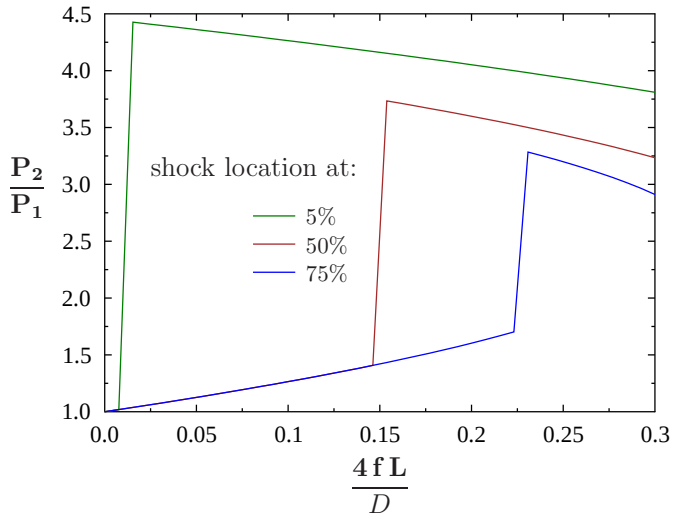


Fig. -10.14. Pressure ratios as a function of $\frac{4fL}{D}$ when the total $\frac{4fL}{D} = 0.3$.

10.9.3 Entrance Mach number, M_1, effects

In this discussion, the effect of changing the throat area on the nozzle efficiency is neglected. In reality these effects have significance and needs to be accounted for some instances. This dissection deals only with the flow when it reaches the supersonic branch reached otherwise the flow is subsonic with regular effects. It is assumed that in this discussion that the pressure ratio $\frac{P_2}{P_1}$ is large enough to create a choked flow and $\frac{4fL}{D}$ is small enough to allow it to occur.

The entrance Mach number, M_1 is a function of the ratio of the nozzle's throat area to the nozzle exit area and its efficiency. This effect is the third parameter discussed here. Practically, the nozzle area ratio is changed by changing the throat area.

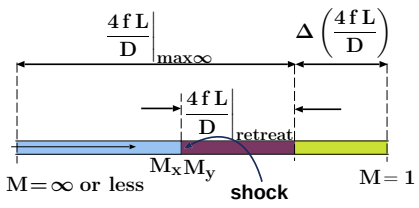


Fig. -10.15. Schematic of a "long" tube in supersonic branch.

As was shown before, there are two different maximums for $\frac{4fL}{D}$; first is the total maximum $\frac{4fL}{D}$ of the supersonic

which depends only on the specific heat, k, and second the maximum depends on the entrance Mach number, M_1. This analysis deals with the case where $\frac{4fL}{D}$ is shorter than total $\frac{4fL}{D}\Big|_{max}$. Obviously, in this situation, the critical point is where $\frac{4fL}{D}$ is equal to $\frac{4fL}{D}\Big|_{max}$ as a result in the entrance Mach number.

The process of decreasing the converging–diverging nozzle's throat increases the entrance[14] Mach number. If the tube contains no supersonic flow then reducing the nozzle throat area wouldn't increase the entrance Mach number.

This part deals with cases where some part of the tube is under supersonic regime and there is a shock as a transition to subsonic branch. Decreasing the nozzle throat area moves the shock location downstream. The "payment" for increase in the supersonic length is by reducing the mass flow. Further, decrease of the throat area results in flushing the shock out of the tube. By doing so, the throat area decreases. The mass flow rate is proportionally linear to the throat area and therefore the mass flow rate reduces. The process of decreasing the throat area also results in increasing the pressure drop of the nozzle (larger resistance in the nozzle[15])[16].

In the case of large tube $\frac{4fL}{D} > \frac{4fL}{D}\Big|_{max}$ the exit Mach number increases with the decrease of the throat area. Once the exit Mach number reaches one no further increases is possible. However, the location of the shock wave approaches to the theoretical location if entrance Mach, $M_1 = \infty$.

The Maximum Location of the Shock

The main point in this discussion however, is to find the furthest shock location downstream. Figure (10.16) shows the possible $\Delta\left(\frac{4fL}{D}\right)$ as a function of retreat of the location of the shock wave from the maximum location. When the entrance Mach number is infinity, $M_1 = \infty$, if the shock location is at the maximum length, then shock at $M_x = 1$ results in $M_y = 1$.

The proposed procedure is based on Figure 10.16.

i) Calculate the extra $\frac{4fL}{D}$ and subtract the actual extra $\frac{4fL}{D}$ assuming shock at the left side (at the max length).

ii) Calculate the extra $\frac{4fL}{D}$ and subtract the actual extra $\frac{4fL}{D}$ assuming shock at the right side (at the entrance).

iii) According to the positive or negative utilizes your root finding procedure.

From numerical point of view, the Mach number equal infinity when left side assumes result in infinity length of possible extra (the whole flow in the tube is subsonic).

[14] The word "entrance" referred to the tube and not to the nozzle. The reference to the tube is because it is the focus of the study here.

[15] Strange? Frictionless nozzle has a larger resistance when the throat area decreases.

[16] It is one of the strange phenomenon that in one way increasing the resistance (changing the throat area) decreases the flow rate while in a different way (increasing the $\frac{4fL}{D}$) does not affect the flow rate.

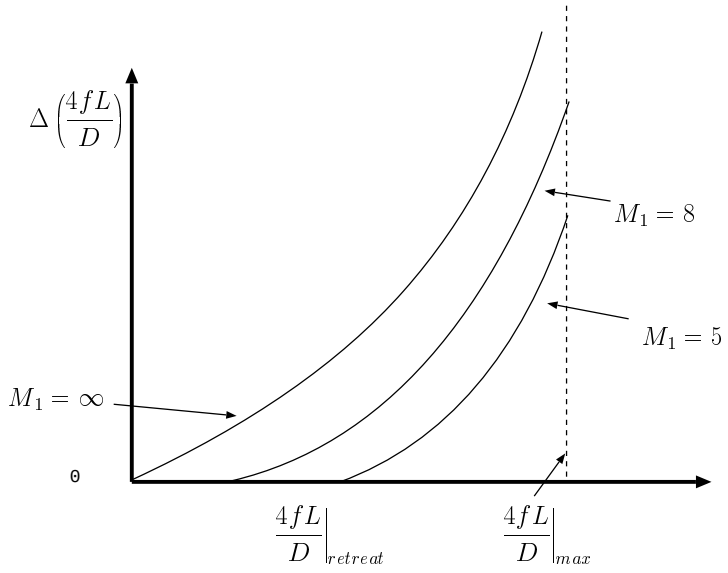


Fig. -10.16. The extra tube length as a function of the shock location, $\frac{4fL}{D}$ supersonic branch.

To overcome this numerical problem, it is suggested to start the calculation from ϵ distance from the right hand side.

Let denote

$$\Delta\left(\frac{4fL}{D}\right) = \frac{4\bar{f}L}{D}_{actual} - \frac{4fL}{D}\Big|_{sup} \tag{10.53}$$

Note that $\frac{4fL}{D}\Big|_{sup}$ is smaller than $\frac{4fL}{D}\Big|_{max_\infty}$. The requirement that has to be satisfied is that denote $\frac{4fL}{D}\Big|_{retreat}$ as difference between the maximum possible of length in which the supersonic flow is achieved and the actual length in which the flow is supersonic see Figure 10.17. The retreating length is expressed as subsonic but

$$\frac{4fL}{D}\Big|_{retreat} = \frac{4fL}{D}\Big|_{max_\infty} - \frac{4fL}{D}\Big|_{sup} \tag{10.54}$$

Figure 10.17 shows the entrance Mach number, M_1 reduces after the maximum length is exceeded.

Example 10.3:
Calculate the shock location for entrance Mach number $M_1 = 8$ and for $\frac{4fL}{D} = 0.9$ assume that $k = 1.4$ ($M_{exit} = 1$).

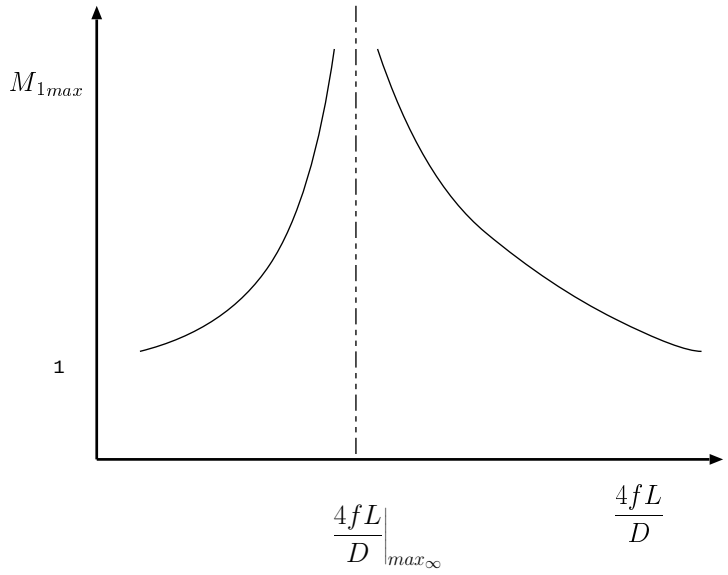


Fig. -10.17. The maximum entrance Mach number, M_1 to the tube as a function of $\frac{4fL}{D}$ supersonic branch.

SOLUTION

The solution is obtained by an iterative process. The maximum $\frac{4fL}{D}\Big|_{max}$ for $k = 1.4$ is 0.821508116. Hence, $\frac{4fL}{D}$ exceed the maximum length $\frac{4fL}{D}$ for this entrance Mach number. The maximum for $M_1 = 8$ is $\frac{4fL}{D} = 0.76820$, thus the extra tube is $\Delta\left(\frac{4fL}{D}\right) = 0.9 - 0.76820 = 0.1318$. The left side is when the shock occurs at $\frac{4fL}{D} = 0.76820$ (flow is choked and no additional $\frac{4fL}{D}$). Hence, the value of left side is -0.1318. The right side is when the shock is at the entrance at which the extra $\frac{4fL}{D}$ is calculated for M_x and M_y is

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
8.0000	0.39289	13.3867	5.5652	74.5000	0.00849

With $(M_1)'$

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.39289	2.4417	2.7461	1.6136	2.3591	0.42390	1.1641

The extra $\Delta\left(\frac{4fL}{D}\right)$ is $2.442 - 0.1318 = 2.3102$ Now the solution is somewhere between the negative of left side to the positive of the right side[17].

In a summary of the actions is done by the following algorithm:

(a) check if the $\frac{4fL}{D}$ exceeds the maximum $\frac{4fL}{D}_{max}$ for the supersonic flow. Accordingly continue.

(b) Guess $\frac{4fL}{D}_{up} = \frac{4fL}{D} - \left.\frac{4fL}{D}\right|_{max}$

(c) Calculate the Mach number corresponding to the current guess of $\frac{4fL}{D}_{up}$,

(d) Calculate the associate Mach number, M_x with the Mach number, M_y calculated previously,

(e) Calculate $\frac{4fL}{D}$ for supersonic branch for the M_x

(f) Calculate the "new and improved" $\frac{4fL}{D}_{up}$

(g) Compute the "new $\frac{4fL}{D}_{down} = \frac{4fL}{D} - \frac{4fL}{D}_{up}$

(h) Check the new and improved $\left.\frac{4fL}{D}\right|_{down}$ against the old one. If it is satisfactory stop or return to stage (b).

Shock location are:

$\mathbf{M_1}$	$\mathbf{M_2}$	$\left.\frac{\mathbf{4fL}}{\mathbf{D}}\right\|_{\mathbf{up}}$	$\left.\frac{\mathbf{4fL}}{\mathbf{D}}\right\|_{\mathbf{down}}$	$\mathbf{M_x}$	$\mathbf{M_y}$
8.0000	1.0000	0.57068	0.32932	1.6706	0.64830

The iteration summary is also shown below

[17] What if the right side is also negative? The flow is chocked and shock must occur in the nozzle before entering the tube. Or in a very long tube the whole flow will be subsonic.

$\mathbf{i}$	$\frac{4fL}{D}\big\vert_{up}$	$\frac{4fL}{D}\big\vert_{down}$	$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{4fL}{D}$
0	0.67426	0.22574	1.3838	0.74664	0.90000
1	0.62170	0.27830	1.5286	0.69119	0.90000
2	0.59506	0.30494	1.6021	0.66779	0.90000
3	0.58217	0.31783	1.6382	0.65728	0.90000
4	0.57605	0.32395	1.6554	0.65246	0.90000
5	0.57318	0.32682	1.6635	0.65023	0.90000
6	0.57184	0.32816	1.6673	0.64920	0.90000
7	0.57122	0.32878	1.6691	0.64872	0.90000
8	0.57093	0.32907	1.6699	0.64850	0.90000
9	0.57079	0.32921	1.6703	0.64839	0.90000
10	0.57073	0.32927	1.6705	0.64834	0.90000
11	0.57070	0.32930	1.6706	0.64832	0.90000
12	0.57069	0.32931	1.6706	0.64831	0.90000
13	0.57068	0.32932	1.6706	0.64831	0.90000
14	0.57068	0.32932	1.6706	0.64830	0.90000
15	0.57068	0.32932	1.6706	0.64830	0.90000
16	0.57068	0.32932	1.6706	0.64830	0.90000
17	0.57068	0.32932	1.6706	0.64830	0.90000

This procedure rapidly converted to the solution.

End Solution

10.10 The Practical Questions and Examples of Subsonic branch

The Fanno is applicable also when the flow isn't choke[18]. In this case, several questions appear for the subsonic branch. This is the area shown in Figure (10.8) in beginning for between points 0 and *a*. This kind of questions made of pair given information to find the conditions of the flow, as oppose to only one piece of information given in choked

[18]These questions were raised from many who didn't find any book that discuss these practical aspects and send the questions to this author.

flow. There many combinations that can appear in this situation but there are several more physical and practical that will be discussed here.

10.10.1 Subsonic Fanno Flow for Given $\frac{4fL}{D}$ and Pressure Ratio

This pair of parameters is the most natural to examine because, in most cases, this information is the only provided information. For a given pipe $\left(\frac{4fL}{D}\right)$, neither the entrance Mach number nor the exit Mach number are given (sometimes the entrance Mach number is given see the next section). There is no known exact analytical solution. There are two possible approaches to solve this problem: one, by building a representative function and find a root (or roots) of this representative function. Two, the problem can be solved by an iterative procedure. The first approach require using root finding method and either method of spline method or the half method or the combination of the two. In the past, this book advocated the integrative method. Recently, this author investigate proposed an improved method.

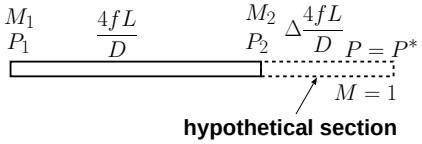


Fig. -10.18. Unchoked flow calculations showing the hypothetical "full" tube when choked

This method is based on the entrance Mach number as the base. The idea based on the idea that the pressure ratio can be drawn as a function of the entrance Mach number. One of difficulties lays in the determination the boundaries of the entrance Mach number. The maximum entrance Mach number is chocking Mach number. The lower possible Mach number is zero which creates very large $\frac{4fL}{D}$. The equations are solve for these large $\frac{4fL}{D}$ numbers by perturbation method and the analytical solution is

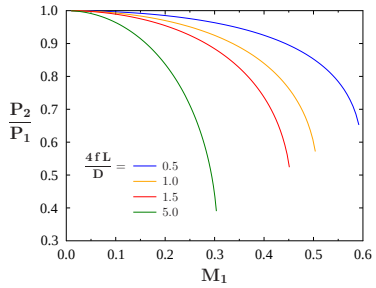


Fig. -10.19. Pressure ratio obtained for a fix $\frac{4fL}{D}$ as a function of Mach number for k=1.4.

$$M_1 = \sqrt{\frac{1 - \left[\frac{P_2}{P_0}\right]^2}{k\,\frac{4fL}{D}}} \qquad (10.55)$$

Equation (10.55) is suggested to be used up to $M_1 < 0.02$. To have small overlapping zone the lower boundary is $M_1 < 0.01$.

The process is based on finding the pressure ratio for given $\frac{4fL}{D}$ pipe dimensionless length. Figure 10.19 exhibits the pressure ratio for fix $\frac{4fL}{D}$ as function of the entrance Mach number. As it can be observed, the entrance Mach number lays between zero

and the maximum of the chocking conditions. For example for a fixed pipe, $\frac{4\,f\,L}{D} = 1$ the maximum Mach number is 0.50874 as shown in Figure 10.19 by orange line. For a given entrance Mach number, the pressure ratio, P_1/P^* and $\left.\frac{4\,f\,L}{D}\right|_1$ can be calculated. The exit pipe length, $\left.\frac{4\,f\,L}{D}\right|_2$ is obtained by subtracting the fix length $\frac{4\,f\,L}{D}$ from $\left.\frac{4\,f\,L}{D}\right|_1$. With this value, the exit Mach number, M_2 and pressure ratio P_2/P^* are calculated. Hence the pressure ratio, P_2/P_1 can be obtained and is drawn in Figure 10.19.

Hence, when the pressure ratio, P_2/P_1 is given along with given pipe, $\frac{4\,f\,L}{D}$ the solution can be obtained by drawing a horizontal line. The intersection of the horizontal line with the right curve of the pressure ratio yields the entrance Mach number. This can be done by a computer program such Potto–GDC (version 0.5.2 and above). The summary of the procedure is as the following.

1) If the pressure ratio is $P_2/P_1 < 0.02$ then using the perturbed solution the entrance Mach number is very small and calculate using the formula

$$M = \sqrt{(1 - \frac{\frac{P_2}{P_1}}{k\left(\frac{4\,f\,L}{D}\right)}} \tag{10.56}$$

If the pressure ratio smaller than continue with the following.

2) Calculate the $\left.\frac{4\,f\,L}{D}\right|_1$ for $M_1 = 0.01$

3) Subtract the given $\frac{4\,f\,L}{D}$ from $\left.\frac{4\,f\,L}{D}\right|_1$ and calculate the exit Mach number.

4) Calculate the pressure ratio.

5) Calculate the pressure ratio for choking condition (given $\frac{4\,f\,L}{D}$.

6) Use your favorite to method to calculate root finding (In potto–GDC Brent's method is used)

Example runs is presented in the Figure 10.20 for $\frac{4\,f\,L}{D} = 0.5$ and pressure ratio equal to 0.8. The blue line in Figure 10.19 intersection with the horizontal line of $P_2/P_1 = 0.8$ yield the solution of $M \sim 0.5$. The whole solution obtained in 7 iterations with accuracy of 10^{-12}.

In Potto–GDC there is another older iterative method used to solve constructed on the properties of several physical quantities must be in a certain range. The first fact is that the pressure ratio P_2/P_1 is always between 0 and 1 (see Figure 10.18). In the figure, a theoretical extra tube is added in such a length that cause the flow to choke (if it really was there). This length is always positive (at minimum is zero).

The procedure for the calculations is as the following:

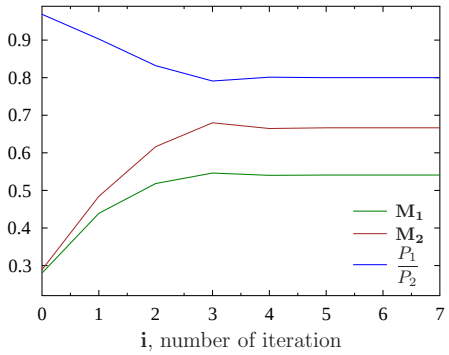


Fig. -10.20. Conversion of solution for given $\frac{4fL}{D} = 0.5$ *and pressure ratio equal 0.8.*

1) Calculate the entrance Mach number, $M_1{}'$ assuming the $\frac{4fL}{D} = \frac{4fL}{D}\Big|_{max}{}'$ (chocked flow);

   Calculate the minimum pressure ratio $(P_2/P_1)_{min}$ for $M_1{}'$ (look at table (10.1))

2) Check if the flow is choked:
   There are two possibilities to check it.

   a) Check if the given $\frac{4fL}{D}$ is smaller than $\frac{4fL}{D}$ obtained from the given P_1/P_2, or
   b) check if the $(P_2/P_1)_{min}$ is larger than (P_2/P_1),

   continue if the criteria is satisfied. Or if not satisfied abort this procedure and continue to calculation for choked flow.

3) Calculate the M_2 based on the $(P^*/P_2) = (P_1/P_2)$,

4) calculate $\Delta\frac{4fL}{D}$ based on M_2,

5) calculate the new (P_2/P_1), based on the new $f\left(\left(\frac{4fL}{D}\right)_1, \left(\frac{4fL}{D}\right)_2\right)$,
   (remember that $\Delta\frac{4fL}{D} = \left(\frac{4fL}{D}\right)_2$),

6) calculate the corresponding M_1 and M_2,

7) calculate the new and "improve" the $\Delta\frac{4fL}{D}$ by

$$\left(\Delta\frac{4fL}{D}\right)_{new} = \left(\Delta\frac{4fL}{D}\right)_{old} * \frac{\left(\frac{P_2}{P_1}\right)_{given}}{\left(\frac{P_2}{P_1}\right)_{old}} \tag{10.57}$$

Note, when the pressure ratios are matching also the $\Delta\frac{4fL}{D}$ will also match.

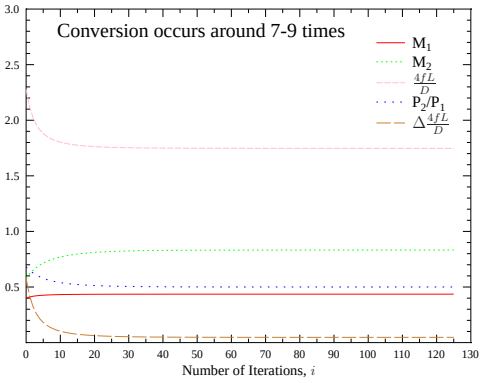
8) Calculate the "improved/new" M_2 based on the improve $\Delta\frac{4fL}{D}$

9) calculate the improved $\frac{4fL}{D}$ as $\frac{4fL}{D} = \left(\frac{4fL}{D}\right)_{given} + \Delta\left(\frac{4fL}{D}\right)_{new}$

10) calculate the improved M_1 based on the improved $\frac{4fL}{D}$.

11) Compare the abs $((P_2/P_1)_{new} - (P_2/P_1)_{old}$) and if not satisfied returned to stage (5) until the solution is obtained.

To demonstrate how this procedure is working consider a typical example of $\frac{4fL}{D} = 1.7$ and $P_2/P_1 = 0.5$. Using the above algorithm the results are exhibited in the following figure.



October 8, 2007

Fig. -10.21. The results of the algorithm showing the conversion rate for unchoked Fanno flow model with a given $\frac{4fL}{D}$ and pressure ratio.

Figure 10.21 demonstrates that the conversion occur at about 7-8 iterations. With better first guess this conversion procedure converts much faster but at certain range it is unstable.

10.10.2 Subsonic Fanno Flow for a Given M_1 and Pressure Ratio

This situation pose a simple mathematical problem while the physical situation occurs in cases where a specific flow rate is required with a given pressure ratio (range) (this problem was considered by some to be somewhat complicated). The specific flow rate can be converted to entrance Mach number and this simplifies the problem. Thus, the problem is reduced to find for given entrance Mach, M_1, and given pressure ratio

calculate the flow parameters, like the exit Mach number, M_2. The procedure is based on the fact that the entrance star pressure ratio can be calculated using M_1. Thus, using the pressure ratio to calculate the star exit pressure ratio provide the exit Mach number, M_2. An example of such issue is the following example that combines also the "Naughty professor" problems.

Example 10.4:
Calculate the exit Mach number for $P_2/P_1 = 0.4$ and entrance Mach number $M_1 = 0.25$.

SOLUTION

The star pressure can be obtained from a table or Potto-GDC as

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.25000	8.4834	4.3546	2.4027	3.6742	0.27217	1.1852

And the star pressure ratio can be calculated at the exit as following

$$\frac{P_2}{P^*} = \frac{P_2}{P_1}\frac{P_1}{P^*} = 0.4 \times 4.3546 = 1.74184$$

And the corresponding exit Mach number for this pressure ratio reads

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.60694	0.46408	1.7418	1.1801	1.5585	0.64165	1.1177

A bit show off the Potto–GDC can carry these calculations in one click as

$\mathbf{M_1}$	$\mathbf{M2}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$
0.25000	0.60693	8.0193	0.40000

End Solution

While the above example show the most simple from of this question, in reality this question is more complicated. One common problem is situation that the diameter is not given but the flow rate and length and pressure (stagnation or static) with some combination of the temperature. The following example deal with one of such example.

Example 10.5:
A tank filled with air at stagnation pressure, $2[Bar]$ should be connected to a pipe with a friction factor, $f = 0.005$, and and length of $5[m]$. The flow rate is (should be)

0.1$\left[\frac{kg}{sec}\right]$ and the static temperature at the entrance of the pipe was measured to be $27°C$. The pressure ratio P_2/P_1 should not fall below 0.9 ($P_2/P_1 > 0.9$). Calculate the exit Mach number, M_2, flow rate, and minimum pipe diameter. You can assume that $k = 1.4$.

SOLUTION

The direct mathematical solution isn't possible and some kind of iteration procedure or root finding for a representative function is needed. For the first part the "naughty professor" procedure cannot be used because $\dot{m}/A$ is not provided and the other hand $\frac{4fL}{D}$ is not provided (missing Diameter). One possible solution is to guess the entrance Mach and check whether and the mass flow rate with the "naughty professor" procedure are satisfied. For Fanno flow for several Mach numbers the following is obtained

$\mathbf{M_1}$	**M2**	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$	**Diameter**
0.100	0.11109	13.3648	0.90000	0.00748
0.150	0.16658	5.8260	0.90000	0.01716
0.200	0.22202	3.1887	0.90000	0.03136

From the last table the diameter can be calculated for example for $M_1 = 0.2$ as

$$D = \frac{4fL}{\frac{4fL}{D}} = 4 \times 0.005 \times 5/3.1887 = 0.03136[m]$$

The same was done for all the other Mach numbers. Now the area can be calculated and therefor the $\dot{m}/A$ can be calculated. With this information the "naughty professor" is given and the entrance Mach number can be calculated. For example for $M_1 = 0.2$ one can obtain the following:

$$\dot{m}/A = 0.1/(\pi \times 0.03136^2/4) \sim 129.4666798$$

The same order as the above table it shown in "naughty professor" (isentropic table).

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
1.5781	0.66752	0.36404	1.2329	0.24300	0.29960	0.56009
0.36221	0.97443	0.93730	1.7268	0.91334	1.5772	0.77785
0.10979	0.99760	0.99400	5.3092	0.99161	5.2647	2.2306

The first result are not reasonable and this process can continue until the satisfactory solution is achieved. Here an graphical approximation is shown (see Figure 10.22). From this exhibit it can be estimated that $M_1 = 0.18$. For this Mach number the following can be obtained

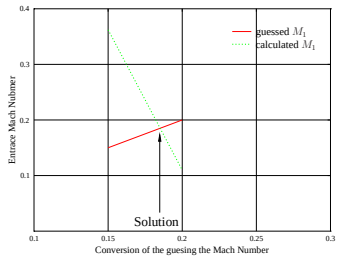


Fig. -10.22. Diagram for finding solution when the pressure ratio and entrance properties (T and P_0 are given.

$\mathbf{M_1}$	**M2**	$\frac{\mathbf{4\,f\,L}}{\mathbf{D}}$	$\frac{\mathbf{P_2}}{\mathbf{P_1}}$
0.18000	0.19985	3.9839	0.90000

Thus, the diameter can be obtained as $D \sim 0.0251[m]$
The flow rate is $\dot{m}/A \sim 202.1[kg/sec \times m^2]$

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.17109	0.99418	0.98551	3.4422	0.97978	3.3726	1.4628

The exact solution is between 0.17 to 0.18 if better accuracy is needed more iterations are required.

End Solution

10.11 The Approximation of the Fanno Flow by Isothermal Flow

The isothermal flow model has equations that theoreticians find easier to use and to compare to the Fanno flow model. One must notice that the maximum temperature at the entrance is T_{01}. When the Mach number decreases the temperature approaches the stagnation temperature ($T \to T_0$). Hence, if one allows certain deviation of temperature, say about 1% that flow can be assumed to be isothermal. This tolerance requires that $(T_0 - T)/T_0 = 0.99$ which requires that enough for $M_1 < 0.15$ even for large $k = 1.67$. This requirement provides that somewhere (depend) in the vicinity of $\frac{4\,f\,L}{D} = 25$ the flow can be assumed isothermal. Hence the mass flow rate is a function of $\frac{4\,f\,L}{D}$ because M_1 changes. Looking at the table or Figure 10.2 or the results from Potto–GDC attached to this book shows that reduction of the mass flow is very rapid.

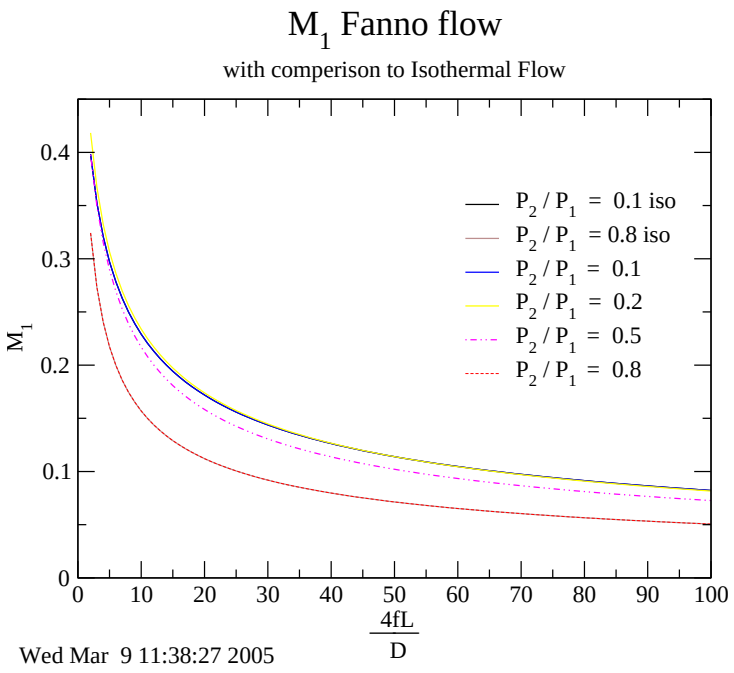


Fig. -10.23. The entrance Mach number as a function of dimensionless resistance and comparison with Isothermal Flow.

As it can be seen for the Figure 10.23 the dominating parameter is $\frac{4\,f\,L}{D}$. The results are very similar for isothermal flow. The only difference is in small dimensionless friction, $\frac{4\,f\,L}{D}$.

10.12 More Examples of Fanno Flow

Example 10.6:
To demonstrate the utility in Figure (10.23) *consider the following example. Find the mass flow rate for* $f = 0.05$, $L = 4[m]$, $D = 0.02[m]$ *and pressure ratio* $P_2/P_1 = 0.1, 0.3, 0.5, 0.8$. *The stagnation conditions at the entrance are* $300K$ *and* $3[bar]$ *air.*

SOLUTION

First calculate the dimensionless resistance, $\frac{4\,f\,L}{D}$.

$$\frac{4\,f\,L}{D} = \frac{4 \times 0.05 \times 4}{0.02} = 40$$

From Figure 10.23 for $P_2/P_1 = 0.1$ $M_1 \approx 0.13$ etc.
or accurately by utilizing the program as in the following table.

$\mathbf{M_1}$	$\mathbf{M_2}$	$\frac{4fL}{D}$	$\frac{4fL}{D}\Big\vert_1$	$\frac{4fL}{D}\Big\vert_2$	$\frac{P_2}{P_1}$
0.12728	1.0000	40.0000	40.0000	0.0	0.11637
0.12420	0.40790	40.0000	42.1697	2.1697	0.30000
0.11392	0.22697	40.0000	50.7569	10.7569	0.50000
0.07975	0.09965	40.0000	107.42	67.4206	0.80000

Only for the pressure ratio of 0.1 the flow is choked.

$\mathbf{M}$	$\frac{T}{T_0}$	$\frac{\rho}{\rho_0}$	$\frac{A}{A^\star}$	$\frac{P}{P_0}$	$\frac{A\times P}{A^*\times P_0}$
0.12728	0.99677	0.99195	4.5910	0.98874	4.5393
0.12420	0.99692	0.99233	4.7027	0.98928	4.6523
0.11392	0.99741	0.99354	5.1196	0.99097	5.0733
0.07975	0.99873	0.99683	7.2842	0.99556	7.2519

Therefore, $T \approx T_0$ and is the same for the pressure. Hence, the mass rate is a function of the Mach number. The Mach number is indeed a function of the pressure ratio but mass flow rate is a function of pressure ratio only through Mach number.

The mass flow rate is

$$\dot{m} = PAM\sqrt{\frac{k}{RT}} = 300000 \times \frac{\pi \times 0.02^2}{4} \times 0.127 \times \sqrt{\frac{1.4}{287300}} \approx 0.48\left(\frac{kg}{sec}\right)$$

and for the rest

$$\dot{m}\left(\frac{\mathbf{P_2}}{\mathbf{P_1}} = 0.3\right) \sim 0.48 \times \frac{0.1242}{0.1273} = 0.468\left(\frac{kg}{sec}\right)$$
$$\dot{m}\left(\frac{\mathbf{P_2}}{\mathbf{P_1}} = 0.5\right) \sim 0.48 \times \frac{0.1139}{0.1273} = 0.43\left(\frac{kg}{sec}\right)$$
$$\dot{m}\left(\frac{\mathbf{P_2}}{\mathbf{P_1}} = 0.8\right) \sim 0.48 \times \frac{0.07975}{0.1273} = 0.30\left(\frac{kg}{sec}\right)$$

End Solution

10.13 The Table for Fanno Flow

Table -10.1. Fanno Flow Standard basic Table k=1.4

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0{}^*}}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U^*}}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$
0.03	787.08	36.5116	19.3005	30.4318	0.03286	1.1998
0.04	440.35	27.3817	14.4815	22.8254	0.04381	1.1996
0.05	280.02	21.9034	11.5914	18.2620	0.05476	1.1994
0.06	193.03	18.2508	9.6659	15.2200	0.06570	1.1991
0.07	140.66	15.6416	8.2915	13.0474	0.07664	1.1988
0.08	106.72	13.6843	7.2616	11.4182	0.08758	1.1985
0.09	83.4961	12.1618	6.4613	10.1512	0.09851	1.1981
0.10	66.9216	10.9435	5.8218	9.1378	0.10944	1.1976
0.20	14.5333	5.4554	2.9635	4.5826	0.21822	1.1905
0.25	8.4834	4.3546	2.4027	3.6742	0.27217	1.1852
0.30	5.2993	3.6191	2.0351	3.0702	0.32572	1.1788
0.35	3.4525	3.0922	1.7780	2.6400	0.37879	1.1713
0.40	2.3085	2.6958	1.5901	2.3184	0.43133	1.1628
0.45	1.5664	2.3865	1.4487	2.0693	0.48326	1.1533
0.50	1.0691	2.1381	1.3398	1.8708	0.53452	1.1429
0.55	0.72805	1.9341	1.2549	1.7092	0.58506	1.1315
0.60	0.49082	1.7634	1.1882	1.5753	0.63481	1.1194
0.65	0.32459	1.6183	1.1356	1.4626	0.68374	1.1065
0.70	0.20814	1.4935	1.0944	1.3665	0.73179	1.0929
0.75	0.12728	1.3848	1.0624	1.2838	0.77894	1.0787
0.80	0.07229	1.2893	1.0382	1.2119	0.82514	1.0638
0.85	0.03633	1.2047	1.0207	1.1489	0.87037	1.0485
0.90	0.01451	1.1291	1.0089	1.0934	0.91460	1.0327
0.95	0.00328	1.061	1.002	1.044	0.95781	1.017
1.00	0.0	1.00000	1.000	1.000	1.00	1.000

Table -10.1. Fanno Flow Standard basic Table (continue)

$\mathbf{M}$	$\frac{\mathbf{4fL}}{\mathbf{D}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho}{\rho^*}$	$\frac{\mathbf{U}}{\mathbf{U}^*}$	$\frac{\mathbf{T}}{\mathbf{T}^*}$
2.00	0.30500	0.40825	1.688	0.61237	1.633	0.66667
3.00	0.52216	0.21822	4.235	0.50918	1.964	0.42857
4.00	0.63306	0.13363	10.72	0.46771	2.138	0.28571
5.00	0.69380	0.089443	25.00	0.44721	2.236	0.20000
6.00	0.72988	0.063758	53.18	0.43568	2.295	0.14634
7.00	0.75280	0.047619	$1.0E+2$	0.42857	2.333	0.11111
8.00	0.76819	0.036860	$1.9E+2$	0.42390	2.359	0.086957
9.00	0.77899	0.029348	$3.3E+2$	0.42066	2.377	0.069767
10.00	0.78683	0.023905	$5.4E+2$	0.41833	2.390	0.057143
20.00	0.81265	0.00609	$1.5E+4$	0.41079	2.434	0.014815
25.00	0.81582	0.00390	$4.6E+4$	0.40988	2.440	0.00952
30.00	0.81755	0.00271	$1.1E+5$	0.40938	2.443	0.00663
35.00	0.81860	0.00200	$2.5E+5$	0.40908	2.445	0.00488
40.00	0.81928	0.00153	$4.8E+5$	0.40889	2.446	0.00374
45.00	0.81975	0.00121	$8.6E+5$	0.40875	2.446	0.00296
50.00	0.82008	0.000979	$1.5E+6$	0.40866	2.447	0.00240
55.00	0.82033	0.000809	$2.3E+6$	0.40859	2.447	0.00198
60.00	0.82052	0.000680	$3.6E+6$	0.40853	2.448	0.00166
65.00	0.82066	0.000579	$5.4E+6$	0.40849	2.448	0.00142
70.00	0.82078	0.000500	$7.8E+6$	0.40846	2.448	0.00122

10.14 Appendix – Reynolds Number Effects

The friction factor in equation (10.24) was assumed constant. In Chapter 9 it was shown that the Reynolds number remains constant for ideal gas fluid. However, in Fanno flow the temperature does not remain constant. Hence, as it was discussed before, the Reynolds number is increasing. Thus, the friction decreases with the exception of the switch in the flow pattern (laminar to turbulent flow). For relatively large relative roughness larger $\epsilon/D > 0.004$ of 0.4% the friction factor is constant. For smoother

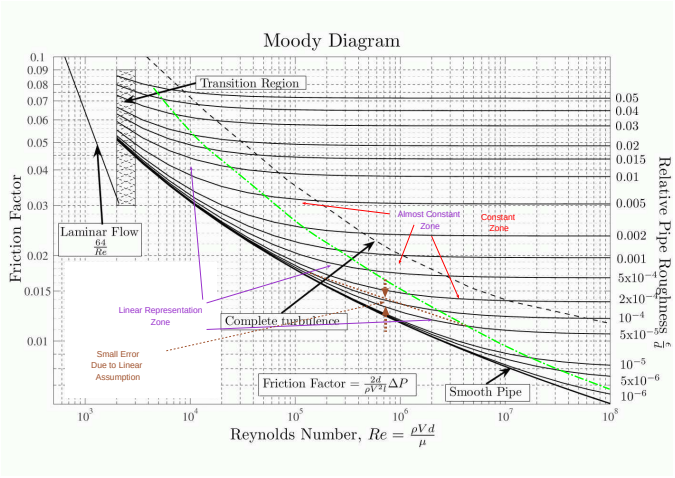


Fig. -10.24. "Moody" diagram on the name of Moody who netscaped H. Rouse's work to claim as his own. In this section, the turbulent area is divided into 3 zones, constant, semi–constant, and a linear After S Beck and R. Collins.

pipe $\epsilon/D < 0.001$ and Reynolds number between 10,000 to a million the friction factor vary between 0.007 to 0.003 with is about factor of two. Thus, the error of $\frac{4fL}{D}$ is limited by a factor of two (2). For this range, the friction factor can be estimated as a linear function of the $log_{10}(Re)$. The error in this assumption is probably small of the assumption that involve in fanno flow model construction. Hence,

$$f = A\, log_{10}(Re) + B \tag{10.58}$$

Where the constant A and B are function of the relative roughness. For most practical purposes the slope coefficient A can be further assumed constant. The slope coefficient $A = -0.998125$. Thus, to carry this calculation relationship between the viscosity and the temperature has to be established. If the viscosity expanded as Taylor or Maclaren series then

$$\frac{\mu}{\mu_1} = A_0 + \frac{A_1\, T}{T_0} + \cdots \tag{10.59}$$

Where μ_1 is the viscosity at the entrance temperature T_1.

Thus, Reynolds number is

$$Re = \frac{D\,\rho\,U}{A_0 + \frac{A_1\, T}{T_0} + \cdots} \tag{10.60}$$

Substituting equation (10.60) into equation (10.58) yield

$$f = A\, log_{10}\left(\frac{D\,\rho\,U}{A_0 + \frac{A_1\, T_2}{T_1} + \cdots}\right) + B \tag{10.61}$$

Left hand side of equation (10.24) is a function of the Mach number since it contains the temperature. If the temperature functionality will not vary similarly to the case of constant friction factor then the temperature can be expressed using equation (10.42).

$$\frac{4}{D}\left(A\,log_{10}\left(\frac{\overbrace{D\,\rho\,U}^{constant}}{A_0+A_1\frac{1+\frac{k-1}{2}{M_1}^2}{1+\frac{k-1}{2}{M_2}^2}+\cdots}\right)+B\right) \tag{10.62}$$

Equation (10.62) is only estimate of the functionally however, this estimate is almost as good as the assumptions of Fanno flow. Equation (10.51) can be improved by using equation (10.62)

$$\frac{4\,L_{max}}{D}\left(A\,log_{10}\left(\frac{\overbrace{D\,\rho\,U}^{constant}}{A_0+A_1\frac{1+\frac{k-1}{2}M^2}{1+\frac{k-1}{2}}}\right)+B\right)\sim\frac{1}{k}\frac{1-M^2}{M^2}+\frac{k+1}{2k}\ln\frac{\frac{k+1}{2}M^2}{1+\frac{k-1}{2}M^2} \tag{10.63}$$

In the most complicate case where the flow pattern is change from laminar flow to turbulent flow the whole Fanno flow model is questionable and will produce poor results.

In summary, in the literature there are three approaches to this issue of non constant friction factor. The friction potential is recommended by a researcher in Germany and it is complicated. The second method substituting this physical approach with numerical iteration. In the numerical iteration method, the expression of the various relationships are inserted into governing differential equations. The numerical methods does not allow flexibility and is very complicated. The methods described here can be expended (if really really needed) and it will be done in very few iteration as it was shown in the Isothermal Chapter.

CHAPTER 11

Rayleigh Flow

Rayleigh flow is a model describing a frictionless flow with heat transfer through a pipe of constant cross sectional area. In practice, Rayleigh flow isn't a really good model to describe real situations. Yet, Rayleigh flow is practical and useful concept in a obtaining trends and limits such as the density and pressure change due to external cooling or heating. As opposed to the two previous models, the heat transfer can be in two directions not like the friction (there is no negative friction). This fact creates a different situation as compared to the previous two models. This model can be applied to cases where the heat transfer is significant and the friction can be ignored. Flow of steam in steam boiler is good examle where Rayleigh flow can be used.

11.1 Introduction

The third simple model for 1–dimensional flow with a constant heat transfer for frictionless flow. This flow is referred to in the literature as Rayleigh Flow (see historical notes). This flow is another extreme case in which the friction effects are neglected because their relative magnitute is significantly smaller than the heat transfer effects. While the isothermal flow model has heat transfer and friction, the main assumption was that relative length is enables significant heat transfer to occur between the surroundings and tube. In contrast, the heat transfer in Rayleigh flow occurs between unknown temperature and the tube and the heat flux is maintained constant. As before, a simple model is built around the

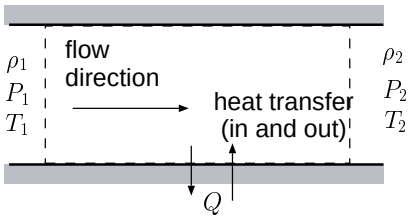


Fig. -11.1. The control volume of Rayleigh Flow.

assumption of constant properties (poorer prediction to case where chemical reaction take a place).

This model is used to roughly predict the conditions which occur mostly in situations involving chemical reaction. In analysis of the flow, one has to be aware that properties do change significantly for a large range of temperatures. Yet, for smaller range of temperatures and lengths the calculations are more accurate. Nevertheless, the main characteristics of the flow such as a choking condition etc. are encapsulated in this model.

The basic physics of the flow revolves around the fact that the gas is highly compressible. The density changes through the heat transfer (temperature change). Contrary to Fanno flow in which the resistance always oppose the flow direction, Rayleigh flow, also, the cooling can be applied. The flow acceleration changes the direction when the cooling is applied.

11.2 Governing Equations

The energy balance on the control volume reads

$$Q = C_p \, (T_{02} - T_{01}) \tag{11.1}$$

The momentum balance reads

$$A \, (P_1 - P_2) = \dot{m} \, (V_2 - V_1) \tag{11.2}$$

The mass conservation reads

$$\rho_1 U_1 A = \rho_2 U_2 A = \dot{m} \tag{11.3}$$

Equation of state

$$\frac{P_1}{\rho_1 \, T_1} = \frac{P_2}{\rho_2 \, T_2} \tag{11.4}$$

There are four equations with four unknowns, if the upstream conditions are known (or downstream conditions are known). Thus, a solution can be obtained. One can notice that equations (11.2), (11.3) and (11.4) are similar to the equations that were solved for the shock wave. Thus, results in the same as before (6.16)

Pressure Ratio

$$\frac{P_2}{P_1} = \frac{1 + k \, {M_1}^2}{1 + k \, {M_2}^2} \tag{11.5}$$

The equation of state (11.4) can further assist in obtaining the temperature ratio as

$$\frac{T_2}{T_1} = \frac{P_2}{P_1} \frac{\rho_1}{\rho_2} \tag{11.6}$$

The density ratio can be expressed in terms of mass conservation as

$$\frac{\rho_1}{\rho_2} = \frac{U_2}{U_1} = \frac{\frac{U_2}{\sqrt{k\,R\,T_2}}\sqrt{k\,R\,T_2}}{\frac{U_1}{\sqrt{k\,R\,T_1}}\sqrt{k\,R\,T_1}} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \tag{11.7}$$

or in simple terms as

Density Ratio

$$\frac{\rho_1}{\rho_2} = \frac{U_2}{U_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \tag{11.8}$$

or substituting equations (11.5) and (11.8) into equation (11.6) yields

$$\frac{T_2}{T_1} = \frac{1 + k\,{M_1}^2}{1 + k\,{M_2}^2}\,\frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \tag{11.9}$$

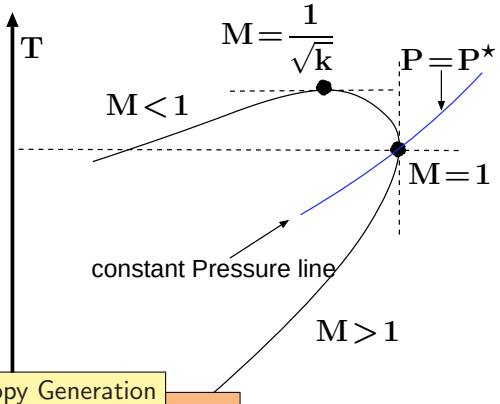
Transferring the temperature ratio to the left hand side and squaring the results gives

Temperature Ratio

$$\frac{T_2}{T_1} = \left[\frac{1 + k\,{M_1}^2}{1 + k{M_2}^2}\right]^2 \left(\frac{M_2}{M_1}\right)^2 \tag{11.10}$$

The Rayleigh line exhibits two possible maximums one for $dT/ds = 0$ and for $ds/dT = 0$. The second maximum can be expressed as $dT/ds = \infty$. The second law is used to find the expression for the derivative.

$$\frac{s_1 - s_2}{C_p} = \ln\frac{T_2}{T_1} - \frac{k-1}{k}\ln\frac{P_2}{P_1} \tag{11.11}$$



Engropy Generation

$$\frac{s_1 - s_2}{C_p} = 2\ln\left[\left(\frac{1 + k\,M_1{}^2)}{(1 + k\,M_2{}^2)}\,\frac{M_2}{M_1}\right] + \frac{k-1}{k}\ln\left[\frac{1 + k\,M21^2}{1 + k\,M_1{}^2}\right] \tag{11.12}$$

Fig. 11.2 The temperature entropy diagram for Rayleigh line.

Let the initial condition M_1, and s_1 be constant and the variable parameters are M_2, and s_2. A derivative of equation (11.12) results in

$$\frac{1}{C_p}\frac{ds}{dM} = \frac{2(1 - M^2)}{M(1 + k\,M^2)} \tag{11.13}$$

Taking the derivative of equation (11.13) and letting the variable parameters be T_2, and M_2 results in

$$\frac{dT}{dM} = constant \times \frac{1 - k\,M^2}{(1 + k\,M^2)^3} \tag{11.14}$$

Combining equations (11.13) and (11.14) by eliminating dM results in

$$\frac{dT}{ds} = constant \times \frac{M(1 - kM^2)}{(1 - M^2)(1 + kM^2)^2} \tag{11.15}$$

On T–s diagram a family of curves can be drawn for a given constant. Yet for every curve, several observations can be generalized. The derivative is equal to zero when $1 - kM^2 = 0$ or $M = 1/\sqrt{k}$ or when $M \to 0$. The derivative is equal to infinity, $dT/ds = \infty$ when $M = 1$. From thermodynamics, increase of heating results in increase of entropy. And cooling results in reduction of entropy. Hence, when cooling is applied to a tube the velocity decreases and when heating is applied the velocity increases. At peculiar point of $M = 1/\sqrt{k}$ when additional heat is applied the temperature decreases. The derivative is negative, $dT/ds < 0$, yet note this point is not the choking point. The choking occurs only when $M = 1$ because it violates the second law. The transition to supersonic flow occurs when the area changes, somewhat similarly to Fanno flow. Yet, choking can be explained by the fact that increase of energy must be accompanied by increase of entropy. But the entropy of supersonic flow is lower (see Figure 11.2) and therefore it is not possible (the maximum entropy at $M = 1$.).

It is convenient to refer to the value of $M = 1$. These values are referred to as the "star"[1] values. The equation (11.5) can be written between choking point and any point on the curve.

Pressure Ratio

$$\frac{P^*}{P_1} = \frac{1 + k\,{M_1}^2}{1 + k} \tag{11.16}$$

The temperature ratio is

Pressure Ratio

$$\frac{T^*}{T_1} = \frac{1}{M^2}\left(\frac{1 + k{M_1}^2}{1 + k}\right)^2 \tag{11.17}$$

The stagnation temperature can be expressed as

$$\frac{T_{01}}{{T_0}^*} = \frac{T_1\left(1 + \dfrac{k-1}{2}{M_1}^2\right)}{T^*\left(\dfrac{1+k}{2}\right)} \tag{11.18}$$

[1] The star is an asterisk.

or explicitly

Stagnation Temperature Ratio

$$\frac{T_{01}}{T_0{}^*} = \frac{2\,(1+k)\,{M_1}^2}{(1+k\,M^2)^2}\left(1+\frac{k-1}{2}{M_1}^2\right) \tag{11.19}$$

The stagnation pressure ratio reads

$$\frac{P_{01}}{P_0{}^*} = \frac{P_1\left(1+\frac{k-1}{2}{M_1}^2\right)}{P^*\left(\frac{1+k}{2}\right)} \tag{11.20}$$

or explicitly

Stagnation Pressure Ratio

$$\frac{P_{01}}{P_0{}^*} = \left(\frac{1+k}{1+k{M_1}^2}\right)\left(\frac{1+k\,{M_1}^2}{\frac{(1+k)}{2}}\right)^{\frac{k}{k-1}} \tag{11.21}$$

11.3 Rayleigh Flow Tables

The "star" values are tabulated in Table 11.1. Several observations can be made in regards to the stagnation temperature. The maximum temperature is not at Mach equal to one. Yet the maximum enetropy accurs at Mach equal to one.

Table -11.1. Rayleigh Flow k=1.4

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0{}^*}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0{}^*}}$	$\frac{\rho^*}{\rho}$
0.03	0.00517	0.00431	2.397	1.267	0.00216
0.04	0.00917	0.00765	2.395	1.266	0.00383
0.05	0.014300	0.011922	2.392	1.266	0.00598
0.06	0.020529	0.017119	2.388	1.265	0.00860
0.07	0.027841	0.023223	2.384	1.264	0.011680
0.08	0.036212	0.030215	2.379	1.262	0.015224
0.09	0.045616	0.038075	2.373	1.261	0.019222
0.10	0.056020	0.046777	2.367	1.259	0.023669
0.20	0.20661	0.17355	2.273	1.235	0.090909
0.25	0.30440	0.25684	2.207	1.218	0.13793
0.30	0.40887	0.34686	2.131	1.199	0.19183

Table -11.1. Rayleigh Flow k=1.4 (continue)

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.35	0.51413	0.43894	2.049	1.178	0.25096
0.40	0.61515	0.52903	1.961	1.157	0.31373
0.45	0.70804	0.61393	1.870	1.135	0.37865
0.50	0.79012	0.69136	1.778	1.114	0.44444
0.55	0.85987	0.75991	1.686	1.094	0.51001
0.60	0.91670	0.81892	1.596	1.075	0.57447
0.65	0.96081	0.86833	1.508	1.058	0.63713
0.70	0.99290	0.90850	1.423	1.043	0.69751
0.75	1.014	0.94009	1.343	1.030	0.75524
0.80	1.025	0.96395	1.266	1.019	0.81013
0.85	1.029	0.98097	1.193	1.011	0.86204
0.90	1.025	0.99207	1.125	1.005	0.91097
0.95	1.015	0.99814	1.060	1.001	0.95693
1.0	1.00	1.00	1.00	1.00	1.000
1.1	0.96031	0.99392	0.89087	1.005	1.078
1.2	0.91185	0.97872	0.79576	1.019	1.146
1.3	0.85917	0.95798	0.71301	1.044	1.205
1.4	0.80539	0.93425	0.64103	1.078	1.256
1.5	0.75250	0.90928	0.57831	1.122	1.301
1.6	0.70174	0.88419	0.52356	1.176	1.340
1.7	0.65377	0.85971	0.47562	1.240	1.375
1.8	0.60894	0.83628	0.43353	1.316	1.405
1.9	0.56734	0.81414	0.39643	1.403	1.431
2.0	0.52893	0.79339	0.36364	1.503	1.455
2.1	0.49356	0.77406	0.33454	1.616	1.475

Table -11.1. Rayleigh Flow k=1.4 (continue)

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
2.2	0.46106	0.75613	0.30864	1.743	1.494
2.3	0.43122	0.73954	0.28551	1.886	1.510
2.4	0.40384	0.72421	0.26478	2.045	1.525
2.5	0.37870	0.71006	0.24615	2.222	1.538
2.6	0.35561	0.69700	0.22936	2.418	1.550
2.7	0.33439	0.68494	0.21417	2.634	1.561
2.8	0.31486	0.67380	0.20040	2.873	1.571
2.9	0.29687	0.66350	0.18788	3.136	1.580
3.0	0.28028	0.65398	0.17647	3.424	1.588
3.5	0.21419	0.61580	0.13223	5.328	1.620
4.0	0.16831	0.58909	0.10256	8.227	1.641
4.5	0.13540	0.56982	0.081772	12.50	1.656
5.0	0.11111	0.55556	0.066667	18.63	1.667
5.5	0.092719	0.54473	0.055363	27.21	1.675
6.0	0.078487	0.53633	0.046693	38.95	1.681
6.5	0.067263	0.52970	0.039900	54.68	1.686
7.0	0.058264	0.52438	0.034483	75.41	1.690
7.5	0.050943	0.52004	0.030094	$1.0E+2$	1.693
8.0	0.044910	0.51647	0.026490	$1.4E+2$	1.695
8.5	0.039883	0.51349	0.023495	$1.8E+2$	1.698
9.0	0.035650	0.51098	0.020979	$2.3E+2$	1.699
9.5	0.032053	0.50885	0.018846	$3.0E+2$	1.701
10.0	0.028972	0.50702	0.017021	$3.8E+2$	1.702
20.0	0.00732	0.49415	0.00428	$1.1E+4$	1.711
25.0	0.00469	0.49259	0.00274	$3.2E+4$	1.712

Table -11.1. Rayleigh Flow k=1.4 (continue)

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
30.0	0.00326	0.49174	0.00190	$8.0E{+}4$	1.713
35.0	0.00240	0.49122	0.00140	$1.7E{+}5$	1.713
40.0	0.00184	0.49089	0.00107	$3.4E{+}5$	1.714
45.0	0.00145	0.49066	0.000846	$6.0E{+}5$	1.714
50.0	0.00117	0.49050	0.000686	$1.0E{+}6$	1.714
55.0	0.000971	0.49037	0.000567	$1.6E{+}6$	1.714
60.0	0.000816	0.49028	0.000476	$2.5E{+}6$	1.714
65.0	0.000695	0.49021	0.000406	$3.8E{+}6$	1.714
70.0	0.000600	0.49015	0.000350	$5.5E{+}6$	1.714

The data is presented in Figure 11.3.

11.4 Examples For Rayleigh Flow

The typical questions that are raised in Rayleigh Flow are related to the maximum heat that can be transferred to gas (reaction heat) and to the maximum flow rate.

Example 11.1:
Air enters a pipe with pressure of $3[bar]$ *and temperature of* $27^{\circ}\mathrm{C}$ *at Mach number of* $M = 0.25$. *Due to internal combustion heat was released and the exit temperature was found to be* $127^{\circ}\mathrm{C}$. *Calculate the exit Mach number, the exit pressure, the total exit pressure, and heat released and transferred to the air. At what amount of energy the exit temperature will start to decrease? Assume* $C_P = 1.004 \left[\frac{kJ}{kg^{\circ}\mathrm{C}}\right]$

Solution

The entrance Mach number and the exit temperature are given and from Table (11.1) or from Potto–GDC the initial ratio can be calculated. From the initial values the ratio at the exit can be computed as the following.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.25000	0.30440	0.25684	2.2069	1.2177	0.13793

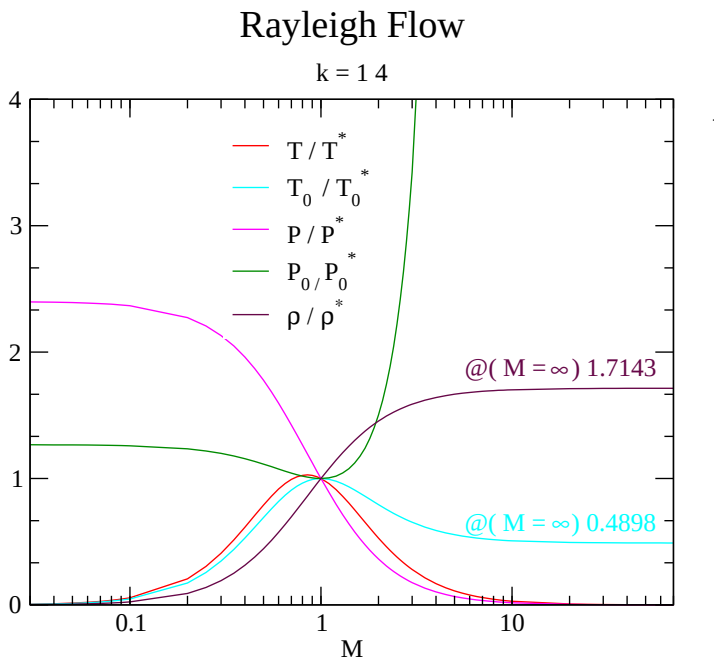


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Fig. -11.3. The basic functions of Rayleigh Flow (k=1.4).

and

$$\frac{T_2}{T^*} = \frac{T_1}{T^*}\frac{T_2}{T_1} = 0.304 \times \frac{400}{300} = 0.4053$$

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0^*}}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0^*}}$	$\frac{\rho^*}{\rho}$
0.29831	0.40530	0.34376	2.1341	1.1992	0.18991

The exit Mach number is known, the exit pressure can be calculated as

$$P_2 = P_1\frac{P^*}{P_1}\frac{P_2}{P^*} = 3 \times \frac{1}{2.2069} \times 2.1341 = 2.901[Bar]$$

For the entrance, the stagnation values are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.25000	0.98765	0.96942	2.4027	0.95745	2.3005	1.0424

The total exit pressure, P_{0_2} can be calculated as the following:

$$P_{0_2} = P_1 \overbrace{\frac{P_{0_1}}{P_1}}^{isentropic} \frac{P_0{}^*}{P_{0_1}} \frac{P_{0_2}}{P_0{}^*} = 3 \times \frac{1}{0.95745} \times \frac{1}{1.2177} \times 1.1992 = 3.08572[Bar]$$

The heat released (heat transferred) can be calculated from obtaining the stagnation temperature from both sides. The stagnation temperature at the entrance, T_{0_1}

$$T_{0_1} = T_1 \overbrace{\frac{T_{0_1}}{T_1}}^{isentropic} = 300/0.98765 = 303.75[K]$$

The isentropic conditions at the exit are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.29831	0.98251	0.95686	2.0454	0.94012	1.9229	0.90103

The exit stagnation temperature is

$$T_{0_2} = T_2 \overbrace{\frac{T_{0_2}}{T_2}}^{isentropic} = 400/0.98765 = 407.12[K]$$

The heat released becomes

$$\frac{Q}{\dot{m}} = C_p \left(T_{0_2} - T_{0_1}\right) 1 \times 1.004 \times (407.12 - 303.75) = 103.78 \left[\frac{kJ}{seckg^\circ\mathrm{C}}\right]$$

The maximum temperature occurs at the point where the Mach number reaches $1/\sqrt{k}$ and at this point the Rayleigh relationship are:

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.84515	1.0286	0.97959	1.2000	1.0116	0.85714

The maximum heat before the temperature can be calculated as following:

$$T_{max} = T_1 \frac{T^*}{T_1} \frac{T_{max}}{T^*} \frac{300}{0.3044} \times 1.0286 = 1013.7[K]$$

The isentropic relationships at the maximum energy are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.84515	0.87500	0.71618	1.0221	0.62666	0.64051	0.53376

The stagnation temperature for this point is

$$T_{0_{max}} = T_{max} * \frac{T_{0_{max}}}{T_{max}} = \frac{1013.7}{0.875} = 1158.51[K]$$

The maximum heat can be calculated as

$$\frac{Q}{\dot{m}} = C_p \left(T_{0_{max}} - T_{0_1}\right) = 1 \times 1.004 \times (1158.51 - 303.75) = 858.18 \left[\frac{kJ}{kgsecK}\right]$$

Note that this point isn't the choking point. After this point additional heat results in temperature reduction.

End Solution

Example 11.2:
Heat is added to the air until the flow is choked in amount of 600 [kJ/kg]. The exit temperature is 1000 [K]. Calculate the entrance temperature and the entrance Mach number.

SOLUTION

The solution involves finding the stagnation temperature at the exit and subtracting the heat (heat equation) to obtain the entrance stagnation temperature. From the Table (11.1) or from the Potto-GDC the following ratios can be obtained.

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
1.0000	0.83333	0.63394	1.0000	0.52828	0.52828	0.52828

The stagnation temperature

$$T_{0_2} = T_2 \frac{T_{0_2}}{T_2} = \frac{1000}{0.83333} = 1200.0[K]$$

The entrance temperature is

$$\frac{T_{0_1}}{T_{0_2}} = 1 - \frac{Q/\dot{m}}{T_{0_2} C_P} = 1200 - \frac{600}{1200 \times 1.004} \cong 0.5016$$

It must be noted that $T_{0_2} = T_0{}^*$. Therefore with $\frac{T_{0_1}}{T_0{}^*} = 0.5016$ either by using Table (11.1) or by Potto-GDC the following is obtained

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.34398	0.50160	0.42789	2.0589	1.1805	0.24362

Thus, entrance Mach number is 0.38454 and the entrance temperature can be calculated as following

$$T_1 = T^* \frac{T_1}{T^*} = 1000 \times 0.58463 = 584.6[K]$$

End Solution

The difference between the supersonic branch to subsonic branch

Example 11.3:
Air with Mach 3 enters a frictionless duct with heating. What is the maximum heat that can be added so that there is no subsonic flow? If a shock occurs immediately at the entrance, what is the maximum heat that can be added?

SOLUTION

To achieve maximum heat transfer the exit Mach number has to be one, $M_2 = 1$.

$$\frac{Q}{\dot{m}} = C_p \left(T_{0_2} - T_{0_1}\right) = C_p {T_0}^* \left(1 - \frac{T_{0_1}}{{T_0}^*}\right)$$

The table for $M = 3$ as follows

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
3.0000	0.28028	0.65398	0.17647	3.4245	1.5882

The higher the entrance stagnation temperature the larger the heat amount that can be absorbed by the flow. In subsonic branch the Mach number after the shock is

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.0000	0.47519	2.6790	3.8571	10.3333	0.32834

With Mach number of $M = 0.47519$ the maximum heat transfer requires information for Rayleigh flow as the following

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.33138	0.47519	0.40469	2.0802	1.1857	0.22844

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.47519	0.75086	0.65398	1.8235	1.1244	0.41176

It also must be noticed that stagnation temperature remains constant across shock wave.

$$\frac{\frac{Q}{\dot{m}}\Big|_{subsonic}}{\frac{Q}{\dot{m}}\Big|_{supersonic}} = \frac{\left(1-\frac{T_{0_1}}{T_0{}^*}\right)_{subsonic}}{\left(1-\frac{T_{0_1}}{T_0{}^*}\right)_{supersonic}} = \frac{1-0.65398}{1-0.65398} = 1$$

It is not surprising for the shock wave to be found in the Rayleigh flow.

End Solution

Example 11.4:

One of the reason that Rayleigh flow model was invented is to be analyzed the flow in a combustion chamber. Consider a flow of air in conduct with a fuel injected into the flow as shown in Figure 11.4. Calculate what the maximum fuel–air ratio. Calculate the exit condition for half the fuel–air ratio. Assume that the mixture properties are of air. Assume that the combustion heat is 25,000[KJ/kg fuel] for the average temperature range for this mixture. Neglect the fuel mass addition and assume that all the fuel is burned (neglect the complications of the increase of the entropy if accrue).

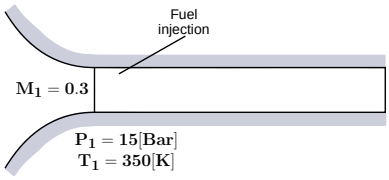


Fig. -11.4. Schematic of the combustion chamber.

SOLUTION

Under these assumptions, the maximum fuel air ratio is obtained when the flow is choked. The entranced condition can be obtained using Potto-GDC as following

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
0.30000	0.40887	0.34686	2.1314	1.1985	0.19183

The choking condition are obtained using also by Potto-GDC as

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{\mathbf{T_0}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{\mathbf{P_0}^*}$	$\frac{\rho^*}{\rho}$
1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

And the isentropic relationships for Mach 0.3 are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
0.30000	0.98232	0.95638	2.0351	0.93947	1.9119	0.89699

The maximum fuel-air can be obtained by finding the heat per unit mass.

$$\frac{\dot{Q}}{\dot{m}} = \frac{Q}{m} = C_p \left(T_{02} - T_{01}\right) = C_p T_1 \left(1 - \frac{T_{01}}{T^*}\right)$$

$$\frac{\dot{Q}}{\dot{m}} = 1.04 \times 350/0.98232 \times (1 - 0.34686) \sim 242.022[kJ/kg]$$

The fuel–air mass ratio has to be

$$\frac{m_{fuel}}{m_{air}} = \frac{\text{needed heat}}{\text{combustion heat}} = \frac{242.022}{25,000} \sim 0.0097[kg\ fuel/kg\ air]$$

If only half of the fuel is supplied then the exit temperature is

$$T_{02} = \frac{Q}{mC_p} + T_{01} = \frac{0.5 \times 242.022}{1.04} + 350/0.98232 \sim 472.656[K]$$

The exit Mach number can be determined from the exit stagnation temperature as following:

$$\frac{T_2}{T^*} = \frac{T_{01}}{{T_0}^*} \frac{T_{02}}{T_{01}}$$

The last temperature ratio can be calculated from the value of the temperatures

$$\frac{T_2}{T^*} = 0.34686 \times \frac{472.656}{350/0.98232} \sim 0.47685$$

The Mach number can be obtained from a Rayleigh table or using Potto-GDC

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T^*}}$	$\frac{\mathbf{T_0}}{{\mathbf{T_0}}^*}$	$\frac{\mathbf{P}}{\mathbf{P^*}}$	$\frac{\mathbf{P_0}}{{\mathbf{P_0}}^*}$	$\frac{\rho^*}{\rho}$
0.33217	0.47685	0.40614	2.0789	1.1854	0.22938

It should be noted that this example is only to demonstrate how to carry the calculations.

End Solution

CHAPTER 12

Evacuating and Filling a Semi Rigid Chambers

In some ways the next two Chapters contain new materials to the traditional compressible flow text books[1]. It was the undersigned experience, that in traditional classes for with compressible flow (sometimes referred to as gas dynamics) don't provide a demonstration to applicability of the class material aside to aeronautical spectrum even such as turbomachinery. In this Chapter a discussion on application of compressible flow to other fields like manufacturing is presented[2].

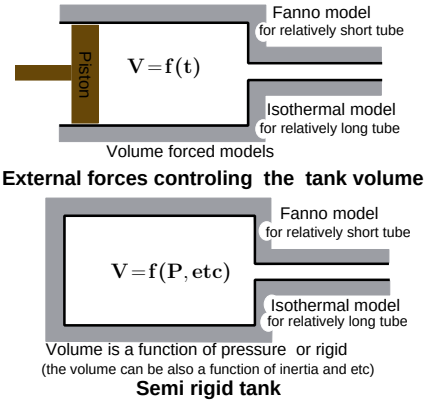


Fig. -12.1. The two different classifications of models that explain the filling or evacuating of a single chamber.

There is a significant importance to the "pure" models such Isothermal flow and Fanno flow which have immediate applicability. However, in many instances, the situations, in life, are far more complicate. Combination of gas compressibility in the chamber and flow out or through a

[1]After completion of these Chapters, the undersigned discover two text books which to include some material related to this topic. These books are OCR, J. A., Fundamentals of Gas Dynamics, International Textbook Co., Scranton, Pennsylvania, 1964. and "Compressible Fluid Flow," 2nd Edition, by M. A. Saad, Prentice Hall, 1985. However, these books contained only limit discussions on the evacuation of chamber with attached nozzle.

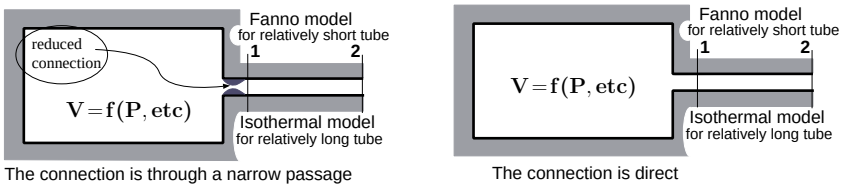
[2]Even if the instructor feels that their students are convinced about the importance of the compressible, this example can further strength and enhance this conviction.

tube post a special interest and these next two Chapters are dealing with these topics. In the first Chapter models, were the chamber volume is controlled or a function of the pressure, are discussed. In the second Chapter, models, were the chamber's volume is a function of external forces, are presented (see Figure (12.1)).

12.1 Governing Equations and Assumptions

The process of filing or evacuating a semi flexible (semi rigid) chamber through a tube is very common in engineering. For example, most car today equipped with an airbag. For instance, the models in this Chapter are suitable for study of the filling the airbag or filling bicycle with air. The analysis is extended to include a semi rigid tank. The term semi rigid tank refers to a tank that the volume is either completely rigid or is a function of the chamber's pressure (or temperature).

As it was shown in this book, the most appropriate model for the flow in the tube for a relatively fast situation is Fanno Flow. The Isothermal model is more appropriate for cases where the tube is relatively long in–which a significant heat transfer occurs keeping the temperature almost constant. As it was shown in Chapter (10) the resistance, $\frac{4fL}{D}$, should be larger than 400. Yet, Isothermal flow model is used as the limiting case.



(a) Reduced connection.

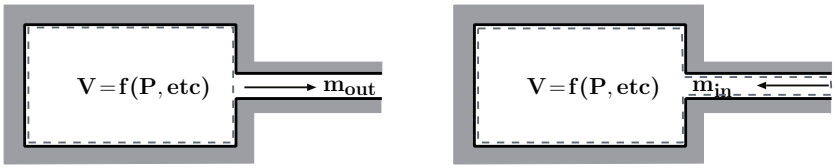
(b) Direct connections.

Fig. -12.2. Comparison direct and reduced connection. These two connection are two limits of the possible connections.

The Rayleigh flow model requires that a constant heat transfer supplied either by chemical reactions or otherwise. This author isn't familiar with situations in which Rayleigh flow model is applicable. And therefore, at this stage, no discussion is offered here.

Fanno flow model is the most appropriate in the case where the filling and evacuating is relatively fast. In case the filling is relatively slow (long $\frac{4fL}{D}$ than the Isothermal flow is appropriate model. Yet as it was stated before, here Isothermal flow and Fanno flow are used as limiting or bounding cases for the real flow. Additionally, the process in the chamber can be limited or bounded between two limits of Isentropic process or Isothermal process.

In this analysis, in order to obtain the essence of the process, some simplified assumptions are made. The assumptions can be relaxed or removed and the model will be more general. Of course, the payment is by far more complex model that sometime



(a) Reduced connection. (b) Direct connections.

Fig. -12.3. Comparison direct and reduced connection. These two connection are two limits of the possible connections.

clutter the physics. First, a model based on Fanno flow model is constructed. Second, model is studied in which the flow in the tube is isothermal. The flow in the tube in many cases is somewhere between the Fanno flow model to Isothermal flow model. This reality is an additional reason for the construction of two models in which they can be compared.

Effects such as chemical reactions (or condensation/evaporation) are neglected. There are two suggested itself possibilities or limits to the connection between the tube to the tank (see the Figure 12.3): one) direct two) through a reduction. The direct connection is when the tube is connect straight or directly to tank like in a case where pipe is welded into the tank. The reduction is typical when a ball is filled trough an one–way valve (filling a baseball ball, also in manufacturing processes[3]). The second possibility leads itself to an additional parameter that is independent of the resistance. The first kind connection tied the resistance, $\frac{4\,f\,L}{D}$, with the tube area.

The simplest model for gas inside the chamber as a first approximation is the isotropic model. It is assumed that kinetic change in the chamber is negligible. This assumption is more suitable for expansion. Therefore, the pressure in the chamber is equal to the stagnation pressure, $P \approx P_0$ (see Figure (12.4)). Thus, the stagnation pressure at the tube's entrance is the same as the pressure in the chamber. It is further can be extended that for a constant compression (constant piston velocity) is also appropriate since the initial shock just change the stagnation temperature and pressure by fix amount. More about this point later.

The mass in the chamber and mass flow out are expressed in terms of the chamber variables (see Figure 12.4). The mass in the tank for perfect gas reads

$$\frac{dm}{dt} - \dot{m}_{out} = 0 \qquad (12.1)$$

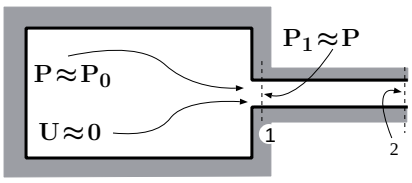


Fig. -12.4. The pressure assumptions in the chamber and tube entrance.

And for perfect gas the mass at any given

[3]For example, in die casting process there is planned reduction to make sure the excess is break at that point.

time is

$$m = \frac{P(t)\,V(t)}{R\,T(t)} \tag{12.2}$$

The mass flow out is a function of the resistance in tube, $\frac{4\,f\,L}{D}$ and the pressure difference between the two sides of the tube $\dot{m}_{out}(\frac{4\,f\,L}{D}, P_1/P_2)$. The initial conditions in the chamber are $T(0)$, $P(0)$ and etc. If the mass occupied in the tube is neglected (only for filling process) the most general equation ideal gas (12.1) reads

$$\frac{d}{dt}\overbrace{\left(\frac{P\,V}{R\,T}\right)}^{m} \pm \overbrace{\rho_1 A \overbrace{c_1 M_1(\frac{4fL}{D}, \frac{P_2}{P_1})}^{U}}^{\dot{m}_{out}} = 0 \tag{12.3}$$

When the plus sign is for filling process and the negative sign is for evacuating process.

12.2 General Model and Non–Dimensionalization

It is convenient to non–dimensioned the properties in chamber by dividing them by their initial conditions. The dimensionless properties of chamber as

$$\overline{T} = \frac{T(t=\bar{t})}{T(t=0)} \tag{12.4a}$$

$$\overline{V} = \frac{V(t=\bar{t})}{V(t=0)} \tag{12.4b}$$

$$\overline{P} = \frac{P(t=\bar{t})}{P(t=0)} \tag{12.4c}$$

$$\bar{t} = \frac{t}{t_c} \tag{12.4d}$$

where t_c is the characteristic time of the system defined as followed

$$t_c = \frac{V(0)}{A\,M_{max}\,\sqrt{k\,R\,T(0))}} \tag{12.5}$$

The physical meaning of characteristic time, t_c is the time which will take to evacuate the chamber if the gas in the chamber was in its initial state, the flow rate was at its maximum (choking flow), and the gas was incompressible in the chamber.

Utilizing these definitions (12.4) and substituting into equation (12.3) yields

$$\frac{P(0)\,V(0)}{t_c\,R\,T(0)}\frac{d}{d\bar{t}}\left(\frac{\overline{P}\,\overline{V}}{\overline{T}}\right) \pm \overbrace{\frac{\overline{P}_1}{R\overline{T}_1}\frac{P(0)}{T(0)}}^{\rho} A\overbrace{\sqrt{k\,R\overline{T}_1 T(0)}}^{c(t)} M_{max}\overline{M}(\bar{t}) = 0 \tag{12.6}$$

where the following definition for the reduced Mach number is added as

$$\overline{M} = \frac{M_1(t)}{M_{max}} \tag{12.7}$$

After some rearranging equation (12.6) obtains the form

$$\frac{d}{d\bar{t}}\left(\frac{\overline{PV}}{\overline{T}}\right) \pm \frac{t_c\, A\, M_{max}\sqrt{k\, R\, T(0)}}{V(0)}\frac{\overline{P_1}\,\overline{M}_1}{\sqrt{\overline{T_1}}}\overline{M} = 0 \tag{12.8}$$

and utilizing the definition of characteristic time, equation (12.5), and substituting into equation (12.8) yields

$$\frac{d}{d\bar{t}}\left(\frac{\overline{P}\,\overline{V}}{\overline{T}}\right) \pm \frac{\overline{P_1}\,\overline{M}}{\sqrt{\overline{T_1}}} = 0 \tag{12.9}$$

Note that equation (12.9) can be modified by introducing additional parameter which referred to as external time, t_{max}[4]. For cases, where the process time is important parameter equation (12.9) transformed to

Governing Equation

$$\frac{d}{d\tilde{t}}\left(\frac{\overline{P}\,\overline{V}}{\overline{T}}\right) \pm \frac{t_{max}}{t_c}\frac{\overline{P_1}\,\overline{M}}{\sqrt{\overline{T_1}}} = 0 \tag{12.10}$$

when $\overline{P}, \overline{V}, \overline{T}$, and $\overline{M}$ are all are function of $\tilde{t}$ in this case. And where $\tilde{t} = t/t_{max}$.

It is more convenient to deal with the stagnation pressure then the actual pressure at the entrance to the tube. Utilizing the equations developed in Chapter 5 between the stagnation condition, denoted without subscript, and condition in a tube denoted with subscript 1. The ratio of $\frac{\overline{P_1}}{\sqrt{\overline{T_1}}}$ is substituted by

$$\frac{\overline{P_1}}{\sqrt{\overline{T_1}}} = \frac{\overline{P}}{\sqrt{\overline{T}}}\left[1 + \frac{k-1}{2}M^2\right]^{\frac{-(k+1)}{2\,(k-1)}} \tag{12.11}$$

It is convenient to denote

$$f[M] = \left[1 + \frac{k-1}{2}M^2\right]^{\frac{-(k+1)}{2\,(k-1)}} \tag{12.12}$$

Note that $f[M]$ is a function of the time. Utilizing the definitions (12.11) and substituting equation (12.12) into equation (12.9) to be transformed into

$$\frac{d}{d\bar{t}}\left(\frac{\overline{P}\,\overline{V}}{\overline{T}}\right) \pm \frac{\overline{P}\,\overline{M}(\bar{t})\, f[M]}{\sqrt{\overline{T}}} = 0 \tag{12.13}$$

[4]This notation is used in many industrial processes where time of process referred to sometime as the maximum time.

Equation (12.13) is a first order nonlinear differential equation that can be solved for different initial conditions. At this stage, the author isn't aware any a general solution for this equation[5]. Nevertheless, many numerical methods are available to solve this equation.

12.2.1 Isentropic Process

The relationship between the pressure and the temperature in the chamber can be approximated as isotropic and therefore

$$\overline{T} = \frac{T(t)}{T(0)} = \left[\frac{P(t)}{P(0)}\right]^{\frac{k-1}{k}} = \overline{P}^{\frac{k-1}{k}} \tag{12.14}$$

The ratios can be expressed in term of the reduced pressure as followed:

$$\frac{\overline{P}}{\overline{\overline{T}}} = \frac{\overline{P}}{\overline{P}^{\frac{k-1}{k}}} = \overline{P}^{\frac{1}{k}} \tag{12.15}$$

and

$$\frac{\overline{P}}{\sqrt{\overline{\overline{T}}}} = \overline{P}^{\frac{k+1}{2\,k}} \tag{12.16}$$

Equation (12.13) is simplified into three different forms:

$$\frac{d}{d\bar{t}}\left(\overline{V}\,\overline{P}^{\frac{1}{k}}\right) \pm \overline{P}^{\frac{k+1}{2\,k}}\,\overline{M}\,(\bar{t})\,f[M] = 0 \tag{12.17a}$$

$$\frac{1}{k}\overline{P}^{\frac{1-k}{k}}\frac{d\overline{P}}{d\bar{t}}\overline{V} + \overline{P}^{\frac{1}{k}}\frac{d\overline{V}}{d\bar{t}} \pm \overline{P}^{\frac{k+1}{2\,k}}\overline{M}(\bar{t})f[M] = 0 \tag{12.17b}$$

$$\overline{V}\frac{d\overline{P}}{d\bar{t}} + k\overline{P}\frac{d\overline{V}}{d\bar{t}} \pm k\overline{P}^{\frac{3\,k-1}{2\,k}}\overline{M}(\bar{t})f[M] = 0 \tag{12.17c}$$

Equation (12.17) is a general equation for evacuating or filling for isentropic process in the chamber. It should be point out that, in this stage, the model in the tube could be either Fanno flow or Isothermal flow. The situations where the chamber undergoes isentropic process but the flow in the tube is Isothermal are limited. Nevertheless, the application of this model provide some kind of a limit where to expect when some heat transfer occurs. Note the temperature in the tube entrance can be above or below the surrounding temperature. Simplified calculations of the entrance Mach number are described in the advance topics section.

[5] To those mathematically included, find the general solution for this equation.

12.2.2 Isothermal Process in The Chamber

12.2.3 A Note on the Entrance Mach number

The value of Mach number, M_1 is a function of the resistance, $\frac{4fL}{D}$ and the ratio of pressure in the tank to the back pressure, P_B/P_1. The exit pressure, P_2 is different from P_B in some situations. As it was shown before, once the flow became choked the Mach number, M_1 is only a function of the resistance, $\frac{4fL}{D}$. These statements are correct for both Fanno flow and the Isothermal flow models. The method outlined in Chapters 9 and 10 is appropriate for solving for entrance Mach number, M_1.

Two equations must be solved for the Mach numbers at the duct entrance and exit when the flow is in a chokeless condition. These equations are combinations of the momentum and energy equations in terms of the Mach numbers. The characteristic equations for Fanno flow (10.51), are

$$\frac{4fL}{D} = \left[\left.\frac{4fL}{D}\right|_{max}\right]_1 - \left[\left.\frac{4fL}{D}\right|_{max}\right]_2 \tag{12.18}$$

and

$$\frac{P_2}{P_0(t)} = \left[1+\frac{k-1}{2}{M_2}^2\right]^{\frac{k}{1-k}} \frac{M_1}{M_2}\sqrt{\left[\frac{1+\frac{k-1}{2}{M_2}^2}{1+\frac{k-1}{2}{M_1}^2}\right]^{\frac{k+1}{k-1}}} \tag{12.19}$$

where $\frac{4fL}{D}$ is defined by equation (10.50).

The solution of equations (12.18) and (12.19) for given $\frac{4fL}{D}$ and $\frac{P_{exit}}{P_0(t)}$ yields the entrance and exit Mach numbers. See advance topic about approximate solution for large resistance, $\frac{4fL}{D}$ or small entrance Mach number, M_1.

12.3 Rigid Tank with A Nozzle

The most simplest possible combination is discussed here before going trough the more complex cases A chamber is filled or evacuated by a nozzle. The gas in the chamber assumed to go an isentropic processes and flow is bounded in nozzle between isentropic flow and isothermal flow[6]. Here, it also will be assumed that the flow in the nozzle is either adiabatic or isothermal.

[6]This work is suggested by Donald Katze the point out that this issue appeared in Shapiro's Book Vol 1, Chapter 4, p. 111 as a question 4.31.

12.3.1 Adiabatic Isentropic Nozzle Attached

The mass flow out is given by either by Fliegner's equation (5.45) or simply use $c\,M\,\rho\,A^*$ and equation (12.17) becomes

$$\frac{1}{k}\overline{P}^{\frac{1-k}{k}}\frac{d\overline{P}}{d\bar{t}} \pm \overline{P}^{\frac{k+1}{2\,k}}(\bar{t})f[M] = 0 \tag{12.20}$$

It was utilized that $\overline{V} = 1$ and $\overline{M}$ definition is simplified as $\overline{M} = 1$. It can be noticed that the characteristic time defined in equation (12.5) reduced into:

$$t_c = \frac{V(0)}{A\sqrt{k\,R\,T(0)}} \tag{12.21}$$

Also it can be noticed that equation (12.12) simplified into

$$f[M] = \left[1 + \frac{k-1}{2}\,1^2\right]^{\frac{-(k+1)}{2(k-1)}} = \left(\frac{k+1}{2}\right)^{\frac{-(k+1)}{2\,(k-1)}} \tag{12.22}$$

Equation (12.20) can be simplified as

$$\frac{1}{k}\left(P^{\frac{1-k}{2\,k}}\right)dP \;\pm\; f[m]\,d\bar{t} = 0 \tag{12.23}$$

Equation (12.23) can be integrated as

$$\int_1^{\overline{P}} P^{\frac{1-k}{2\,k}}\,dP \pm \int_0^t dt = 0 \tag{12.24}$$

The integration limits are obtained by simply using the definitions of reduced pressure, at $P(\bar{t}=0) = 1$ and $P(\bar{t}=\bar{t}) = \overline{P}$. After the integration, equation (12.24) and rearrangement becomes

$$\overline{P} = \left[1 \pm \left(\frac{k-1}{2}\right)f[M]\bar{t}\right]^{\frac{2\,k}{1-k}} \tag{12.25}$$

Example 12.1:
A chamber is connected to a main line with pressure line with a diaphragm and nozzle. The initial pressure at the chamber is $1.5[Bar]$ *and the volume is* $1.0[m^3]$*. Calculate time it requires that the pressure to reach 5[Bar] for two different nozzles throat area of 0.001, and 0.1* $[m^2]$ *when diaphragm is erupted. Assumed the stagnation temperature at the main line is the ambient of* $27[^\circ\mathrm{C}]$.

SOLUTION

The characteristic time is

$$t_{max} = \frac{V}{A^*\,c} = \frac{V}{A^*c} = \frac{1.0}{0.1\sqrt{1.4\times 287\times 300}} = 0.028[sec] \tag{12.26}$$

And for smaller area

$$t_{max} = \frac{1.0}{0.001\sqrt{1.4 \times 287 \times 300}} = 2.8[sec]$$

$$\overline{P} = \frac{P(t)}{P(0)} = \frac{4.5}{1.5} = 3.0$$

The time is

$$t = t_{max}\left[\overline{P}^{\frac{1-k}{k}} - 1\right]\left(\frac{k+1}{2}\right)^{-()} \tag{12.27}$$

Substituting values into equation (12.27) results

$$t = 0.028\left[3^{\frac{1-1.4}{2.8}} - 1\right]\left(\frac{2.4}{2}\right)^{\frac{-2.4}{0.8}} = 0.013[sec] \tag{12.28}$$

End Solution

Filling/Evacuating The Chamber Under choked Condition

The flow in the nozzle can became unchoked and it can be analytically solved. Owczarek [1964] found an analytical solution which described here.

12.3.2 Isothermal Nozzle Attached

In this case the process in nozzle is assumed to isothermal but the process in the chamber is isentropic. The temperature in the nozzle is changing because the temperature in the chamber is changing. Yet, the differential temperature change in the chamber is slower than the temperature change in nozzle. For rigid volume, $\overline{V} = 1$ and for isothermal nozzle $\overline{T} = 1$ Thus, equation (12.13) is reduced into

$$\frac{d\overline{P}}{d\bar{t}} = \pm f[M]\overline{P} = 0 \tag{12.29}$$

Separating the variables and rearranging equation (12.29) converted into

$$\int_1^{\overline{P}} \frac{d\overline{P}}{\overline{P}} \pm f[M]\int_0^{\bar{t}} d\bar{t} = 0 \tag{12.30}$$

Here, $f[M]$ is expressed by equation (12.22). After the integration, equation (12.30) transformed into

$$\ln \overline{P} = \left(\frac{k+1}{2}\right)^{\frac{-(k+1)}{2(k-1)}} \bar{t}$$

$$\overline{P} = e^{\left[\left(\frac{k+1}{2}\right)^{\frac{-(k+1)}{2(k-1)}}\bar{t}\right]} \tag{12.31}$$

12.4 Rapid evacuating of a rigid tank

12.4.1 Assuming Fanno Flow Model

The relative Volume, $\overline{V}(t) = 1$, is constant and equal one for a completely rigid tank. In such case, the general equation (12.17) "shrinks" and doesn't contain the relative volume term.

A reasonable model for the tank is isentropic (can be replaced polytropic relationship) and Fanno flow are assumed for the flow in the tube. Thus, the specific governing equation is

$$\frac{d\overline{P}}{d\bar{t}} - k\overline{M} f[M] \overline{P}^{\frac{3\,k-1}{2\,k}} = 0 \tag{12.32}$$

For a choked flow the entrance Mach number to the tube is at its maximum, M_{max} and therefore $\overline{M} = 1$. The solution of equation (12.32) is obtained by noticing that $\overline{M}$ is not a function of time and by variables separation results in

$$\int_0^{\bar{t}} d\bar{t} = \int_1^{\overline{P}} \frac{d\overline{P}}{k\overline{M} f[M] \overline{P}^{\frac{3\,k-1}{2\,k}}} = \frac{1}{k\overline{M} f[M]} \int_1^{\overline{P}} \overline{P}^{\frac{1-3\,k}{2\,k}} d\overline{P} \tag{12.33}$$

direct integration of equation (12.33) results in

$$\bar{t} = \frac{2}{(k-1)\overline{M}\, f[M]} \left[\overline{P}^{\frac{1-k}{2\,k}} - 1 \right] \tag{12.34}$$

It has to be realized that this is "reversed" function i.e. $\bar{t}$ is a function of P and can be reversed for case. But for the chocked case it appears as

$$\overline{P} = \left[1 + \frac{(k-1)\overline{M} f[M]}{2} \bar{t} \right]^{\frac{2\,k}{1-k}} \tag{12.35}$$

The function is drawn as shown here in Figure (12.5).

The Figure (12.5) shows that when the modified reduced pressure equal to one the reduced time is zero. The reduced time increases with decrease of the pressure in the tank.

At certain point the flow becomes chokeless flow (unless the back pressure is complete vacuum). The transition point is denoted here as chT. Thus, equation (12.34) has to include the entrance Mach under the integration sign as

$$\bar{t} - \bar{t}_{chT} = \int_{P_{chT}}^{\overline{P}} \frac{1}{k\overline{M} f[M]} \overline{P}^{\frac{1-3\,k}{2\,k}} d\overline{P} \tag{12.36}$$

For practical purposes if the flow is choked for more than 30% of the characteristic time the choking equation can be used for the whole range, unless extra long time or extra low pressure is calculated/needed. Further, when the flow became chokeless the entrance Mach number does not change much from the choking condition.

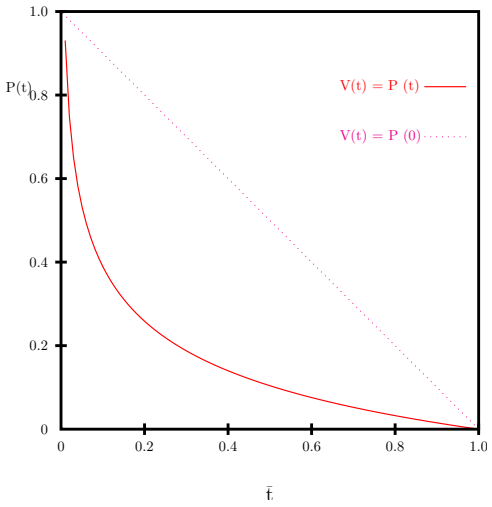


Fig. -12.5. The reduced time as a function of the modified reduced pressure

Again, for the special cases where the choked equation is not applicable the integration has to be separated into zones: choked and chokeless flow regions. And in the choke region the calculations can use the choking formula and numerical calculations for the rest.

Example 12.2:
A chamber with volume of 0.1[m^3] is filled with air at pressure of 10[Bar]. The chamber is connected with a rubber tube with $f = 0.025$, $d = 0.01[m]$ and length of $L = 5.0[m]$

SOLUTION

The first parameter that calculated is $\frac{4\,f\,L}{D} = 5$

End Solution

12.4.2 Filling Process

The governing equation is

$$\frac{d\overline{P}}{d\bar{t}} - k\overline{M}f[M]\overline{P}^{\frac{3\,k-1}{2\,k}} = 0 \tag{12.37}$$

For a choked flow the entrance Mach number to the tube is at its maximum, M_{max} and therefore $\overline{M} = 1$. The solution of equation (12.37) is obtained by noticing that $\overline{M}$

is not a function of time and by variable separation results in

$$\int_0^{\overline{t}} d\overline{t} = \int_1^{\overline{P}} \frac{d\overline{P}}{k\overline{M}f[M]\overline{P}^{\frac{3k-1}{2k}}} = \frac{1}{k\overline{M}f[M]} \int_1^{\overline{P}} \overline{P}^{\frac{1-3k}{2k}} d\overline{P} \tag{12.38}$$

direct integration of equation (12.38) results in

$$\overline{t} = \frac{2}{(k-1)\overline{M}f[M]} \left[\overline{P}^{\frac{1-k}{2k}} - 1 \right] \tag{12.39}$$

It has to be realized that this is a reversed function. Nevertheless, with today computer this should not be a problem and easily can be drawn as shown here in Figure (12.5). The Figure shows that when the modified reduced pressure equal to one the

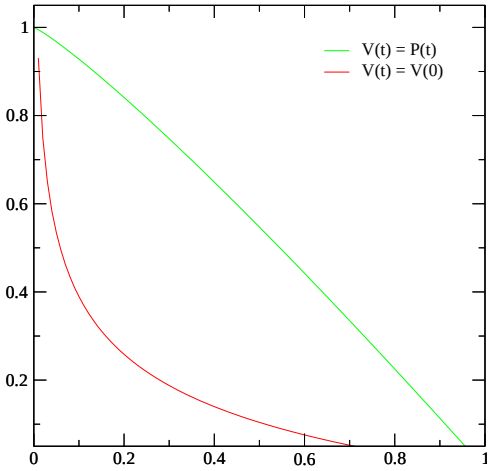


Fig. -12.6. The reduced time as a function of the modified reduced pressure

reduced time is zero. The reduced time increases with decrease of the pressure in the tank.

At some point the flow becomes chokeless flow (unless the back pressure is a complete vacuum). The transition point is denoted here as chT. Thus, equation (12.39) has to include the entrance Mach under the integration sign as

$$\overline{t} - \overline{t}_{chT} = \int_{P_{chT}}^{\overline{P}} \frac{1}{k\overline{M}f[M]} \overline{P}^{\frac{1-3k}{2k}} d\overline{P} \tag{12.40}$$

12.4.3 The Isothermal Process

For Isothermal process, the relative temperature, $\overline{T} = 1$. The combination of the isentropic tank and Isothermal flow in the tube is different from Fanno flow in that the

chocking condition occurs at $1/\sqrt{k}$. This model is reasonably appropriated when the chamber is insulated and not flat while the tube is relatively long and the process is relatively long.

It has to be remembered that the chamber can undergo isothermal process. For the double isothermal (chamber and tube) the equation (12.6) reduced into

$$\frac{P(0)V(0)}{t_c R T(0)}\frac{d(\overline{PV})}{d\bar{t}} \pm \overbrace{\frac{\overline{P}_1}{R}\frac{P(0)}{T(0)}}^{\rho} A\overbrace{\sqrt{k\,R\,T(0)}}^{c(0)} M_{max}\overline{M}(\bar{t}) = 0 \tag{12.41}$$

12.4.4 Simple Semi Rigid Chamber

A simple relation of semi rigid chamber when the volume of the chamber is linearly related to the pressure as

$$V(t) = a\,P(t) \tag{12.42}$$

where a is a constant that represent the physics. This situation occurs at least in small ranges for airbag balloon etc. The physical explanation when it occurs beyond the scope of this book. Nevertheless, a general solution is easily can be obtained similarly to rigid tank. Substituting equation (12.42) into yields

$$\frac{d}{d\bar{t}}\left(\overline{P}^{\frac{1+k}{k}}\right) - \overline{P}^{\frac{k+1}{2\,k}}\,\overline{M}\,f[M] = 0 \tag{12.43}$$

Carrying differentiation result in

$$\frac{1+k}{k}\overline{P}^{\frac{1}{k}}\frac{d\overline{P}}{d\bar{t}} - \overline{P}^{\frac{k+1}{2\,k}}\overline{M}f[M] = 0 \tag{12.44}$$

Similarly as before, the variables are separated as

$$\int_0^{\bar{t}} dt = \frac{k}{1+k}\int_1^{\overline{P}} \frac{\overline{P}^{\frac{k-1}{2\,k}}d\overline{P}}{\overline{M}f[M]} \tag{12.45}$$

The equation (12.45) integrated to obtain the form

$$\bar{t} = \frac{2\,k^2}{\overline{M}f[M](3\,k-1)(1+k)}\left[1 - \overline{P}^{\frac{3\,k-1}{2\,k}}\right] \tag{12.46}$$

The physical meaning that the pressure remains larger thorough evacuating process, as results in faster reduction of the gas from the chamber.

12.4.5 The "Simple" General Case

The relationship between the pressure and the volume from the physical point of view must be monotonous. Further, the relation must be also positive, increase of the

pressure results in increase of the volume (as results of Hook's law. After all, in the known situations to this author pressure increase results in volume decrease (at least for ideal gas.).

In this analysis and previous analysis the initial effect of the chamber container inertia is neglected. The analysis is based only on the mass conservation and if unsteady effects are required more terms (physical quantities) have taken into account. Further, it is assumed the ideal gas applied to the gas and this assumption isn't relaxed here.

Any continuous positive monotonic function can be expressed into a polynomial function. However, as first approximation and simplified approach can be done by a single term with a different power as

$$V(t) = aP^n \tag{12.47}$$

When n can be any positive value including zero, 0. The physical meaning of $n = 0$ is that the tank is rigid. In reality the value of n lays between zero to one. When n is approaching to zero the chamber is approaches to a rigid tank and vis versa when the $n \to 1$ the chamber is flexible like a balloon.

There isn't a real critical value to n. Yet, it is convenient for engineers to further study the point where the relationship between the reduced time and the reduced pressure are linear[7] Value of n above it will Convex and below it concave.

$$\frac{d}{d\overline{t}}\left(\overline{P}^{\frac{1+nk-k}{k}}\right) - \overline{P}^{\frac{k+1}{2\,k}}\,\overline{M}f[M] = 0 \tag{12.48}$$

Notice that when $n = 1$ equation (12.49) reduced to equation (12.43).

After carrying–out differentiation results

$$\frac{1+nk-k}{k}\overline{P}^{\frac{1+nk-2\,k}{k}}\frac{d\overline{P}}{d\overline{t}} - \overline{P}^{\frac{k+1}{2\,k}}\,\overline{M}f[M] = 0 \tag{12.49}$$

Again, similarly as before, variables are separated and integrated as follows

$$\int_0^{\overline{t}} dt = \frac{1+nk-k}{k}\int_1^{\overline{P}} \frac{\overline{P}^{\frac{1+2nk-5\,k}{2\,k}}d\overline{P}}{\overline{M}f[M]} \tag{12.50}$$

Carrying–out the integration for the initial part if exit results in

$$\overline{t} = \frac{2\,k^2}{\overline{M}f[M](3\,k-2nk-1)(1+k)}\left[1-\overline{P}^{\frac{3\,k-2nk-1}{2\,k}}\right] \tag{12.51}$$

The linear condition are obtain when

$$3\,k-2nk-1 = 1 \longrightarrow n = \frac{3\,k-2}{2\,k} \tag{12.52}$$

That is just bellow 1 ($n = 0.785714286$) for $k = 1.4$.

[7] Some suggested this border point as infinite evocation to infinite time for evacuation etc. This undersigned is not aware situation where this indeed play important role. Therefore, it is waited to find such conditions before calling it as critical condition.

12.5 Advance Topics

The term $\frac{4fL}{D}$ is very large for small values of the entrance Mach number which requires keeping many digits in the calculation. For small values of the Mach numbers, equation (12.18) can be approximated as

$$\frac{4\,f\,L}{D} = \frac{1}{k}\,\frac{{M_{exit}}^2 - {M_{in}}^2}{{M_{exit}}^2 {M_{in}}^2} \tag{12.53}$$

and equation (12.19) as

$$\frac{P_{exit}}{P_0(t)} = \frac{M_{in}}{M_{exit}}. \tag{12.54}$$

The solution of two equations (12.53) and (12.54) yields

$$M_{in} = \sqrt{\frac{1 - \left[\frac{P_{exit}}{P_0(t)}\right]^2}{k\,\frac{4\,f\,L}{D}}} \tag{12.55}$$

This solution should used only for $M_{in} < 0.00286$; otherwise equations (12.18) and (12.19) must be solved numerically[8].

The solution of equation (12.18) and (12.19) is described in "Pressure die casting: a model of vacuum pumping" Bar-Meir, G; Eckert, E R G; Goldstein, R. J. Journal of Manufacturing Science and Engineering (USA). Vol. 118, no. 2, pp. 259-265. May 1996.

12.6 Remark on Real Gases

In the above discussion was dealing with ideal gas[9]. The real gas equation is used in some situations to describe situations were the deviations from the ideal gas model are significant. The equation of the state is than (see equation (2.39))

$$z = \frac{P}{\rho\,R\,T} \tag{12.56}$$

The mass in the chamber is

$$m = \frac{P(t)\,V(t)}{R\,z(t)\,T(t)} \tag{12.57}$$

Thus the change with respect to time for the mass in the chamber is

$$\frac{dm}{dt} = \frac{1}{R}\left(\frac{d}{dt}\left(\frac{P(t)\,V(t)}{T(t)}\right)\,\frac{1}{z(t)} - \frac{1}{z^2}\,\left(\frac{dz}{d\rho}\frac{d\rho}{dt} + \frac{dz}{dP}\frac{dP}{dt} + \frac{dz}{dT}\frac{dT}{dt}\right)\right) \tag{12.58}$$

[8] When the accuracy is less significant but speed is important it range can be increased

[9] This point was raised by several people in Henry Foust from Texas, John McReaimie from Canada and Xunfei Zhou from Xian Jiaotong University China

The flow out using equation (4.37) is

$$\dot{m}_{out} = A\,c\,M\,\rho = A\sqrt{z\,n\,R\,T} \tag{12.59}$$

The process is assumed to be isentropic hence equation (4.36) represent the relationship

$$\left(\frac{\rho_1}{\rho_0}\right)^n = \frac{P_1}{P_0} \tag{12.60}$$

Also the relationship density and the temperature is

$$\left(\frac{\rho_1}{\rho_0}\right)^{n-1} = \frac{z_1\,T_1}{z_0\,T_0} \tag{12.61}$$

and relationship between the pressure and temperature is

$$\left(\frac{P_1}{P_0}\right)^{1-n} = \frac{z_0\,T_0}{z_1\,T_1} \tag{12.62}$$

With the terms that were developed earlier such equation (5.155) provide the based to build the model for real gases.

CHAPTER 13

Evacuating/Filing Chambers under External Volume Control

This chapter is the second on the section dealing with filling and evacuating chambers. Here the model deals with the case where the volume is controlled by external forces. This kind of model is applicable to many manufacturing processes such as die casting, extraction etc. In general the process of the displacing the gas (in many cases air) with a liquid is a very common process. For example, in die casting process liquid metal is injected to a cavity and after the cooling/solidification period a part is obtained in near the final shape. One can also view the exhaust systems of internal combustion engine in the same manner. In these processes, sometime is vital to obtain a proper evacuation of the gas (air) from the cavity.

13.1 General Model

In this analysis, in order to obtain the essence of the process, some simplified assumptions are made. It simplest model of such process is when a piston is displacing the gas though a long tube. It assumed that no chemical reaction (or condensation/evaporation) occur in the piston or the tube [1]. It is further assumed that the process is relatively fast. The last assumption is a appropriate assumption in process such as die casting.

Two extreme possibilities again suggest themselves: rapid and slow processes. The two different connections, direct and through reduced area are combined in this analysis.

[1] such reaction are possible and expected to be part of process but the complicates the analysis and do not contribute to understand to the compressibility effects.

13.1.1 Rapid Process

Clearly under the assumption of rapid process the heat transfer can be neglected and Fanno flow can be assumed for the tube. The first approximation isotropic process describe the process inside the cylinder (see Figure 13.1).

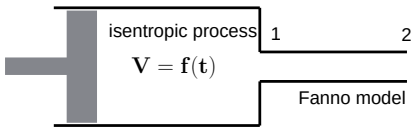


Fig. -13.1. The control volume of the "Cylinder."

Before introducing the steps of the analysis, it is noteworthy to think about the process in qualitative terms. The replacing incompressible liquid enter in the same amount as replaced incompressible liquid. But in a compressible substance the situation can be totally different, it is possible to obtain a situation where that most of the liquid entered the chamber and yet most of the replaced gas can be still be in the chamber. Obtaining conditions where the volume of displacing liquid is equal to the displaced liquid are called the critical conditions. These critical conditions are very significant that they provide guidelines for the design of processes.

Obviously, the best ventilation is achieved with a large tube or area. In manufacture processes to minimize cost and the secondary machining such as trimming and other issues the exit area or tube has to be narrow as possible. In the exhaust system cost of large exhaust valve increase with the size and in addition reduces the strength with the size of valve[2]. For these reasons the optimum size is desired. The conflicting requirements suggest an optimum area, which is also indicated by experimental studies and utilized by practiced engineers.

The purpose of this analysis to yield a formula for critical/optimum vent area in a simple form. The second objective is to provide a tool to "combine" the actual tube with the resistance in the tube, thus, eliminating the need for calculations of the gas flow in the tube to minimize the numerical calculations.

A linear function is the simplest model that decibels changes the volume. Later a discussion on the non-linear functions on effect of the linear function. In reality, in some situations like die casting this description is appropriate. Nevertheless, this model can be extended numerical in cases where more complex function is applied.

$$V(t) = V(0)\left[1 - \frac{t}{t_{max}}\right] \tag{13.1}$$

Equation (13.1) can be non–dimensionlassed as

$$\overline{V}(\bar{t}) = 1 - \bar{t} \tag{13.2}$$

The governing equation (12.10) which was developed in the previous Chapter

[2]After certain sizes, the possibility of crack increases.

12 obtained the form as

Piston Continuity

$$\left[\overline{P}\right]^{\frac{1}{k}} \left\{ \frac{1}{k} \frac{\overline{V}}{\overline{P}} \frac{d\overline{P}}{dt} + \frac{d\overline{V}}{d\bar{t}} \right\} + \frac{t_{max}\, \overline{M}\, f(M)}{t_c} \left[\overline{P}\right]^{\frac{k+1}{2\,k}} = 0 \tag{13.3}$$

where $\bar{t} = t/t_{max}$. Notice that in this case that there are two different characteristic times: the "characteristic" time, t_c and the "maximum" time, t_{max}. The first characteristic time, t_c is associated with or related to the ratio of the volume and the tube characteristics (see equation (12.5)). The second characteristic time, t_{max} is associated with the imposed time on the system (in this case the elapsed time of the piston stroke) or the time to cycle period for repetitive process.

Equation (13.3) is an nonlinear first order differential equation and can be rearranged as follows

$$\frac{d\overline{P}}{k \left(1 - \frac{t_{max}}{t_c} \overline{M}\, f[M] \overline{P}^{\frac{k-1}{2\,k}} \right) \overline{P}} = \frac{d\bar{t}}{1-\bar{t}} \qquad ; \qquad \overline{P}(0) = 1. \tag{13.4}$$

Equation (13.4) is can be solved only when the flow is chocked. In which case, $f[m]$ isn't function of the time.

The solution of equation (13.4)) can be obtained by transforming and by introducing a new variable $\xi = \overline{P}^{\frac{k-1}{2\,k}}$ and therefore $\overline{P} = [\xi]^{\frac{2\,k}{k-1}}$. The reduced Pressure derivative, $d\overline{P} = \frac{2\,k}{k-1}\, [\xi]^{\left(\frac{2\,k}{k-1}\right)-1}\, d\xi$ Utilizing this definition and there implication reduce equation (13.4)

$$\frac{2\, [\xi]^{\left(\frac{2\,k}{k-1}\right)-1}\, d\xi}{(k-1)\, (1-B\xi)\, [\xi]^{\frac{2\,k}{k-1}}} = \frac{d\bar{t}}{1-\bar{t}} \tag{13.5}$$

where $B = \frac{t_{max}}{t_c}\, \overline{M}\, f[M]$ And equation (13.5) can be further simplified as

$$\frac{2\, d\xi}{(k-1)\,(1-B\xi)\,\xi} = \frac{d\bar{t}}{1-\bar{t}} \tag{13.6}$$

Equation (13.6) can be integrated to obtain

$$\frac{2}{(k-1)\, B}\, \ln \left| \frac{1-B\,\xi}{\xi} \right| = -\ln \bar{t} \tag{13.7}$$

or in a different form

$$\left| \frac{1-B\xi}{\xi} \right|^{\frac{2}{(1-k)\, B}} = \bar{t} \tag{13.8}$$

Now substituting to the "preferred" variable

$$\left[\frac{1-\frac{t_{max}}{t_c}\,\overline{M}\,f[M]\,\overline{P}^{\frac{k-1}{2\,k}}}{\overline{P}^{\frac{k-1}{2\,k}}}\right]\frac{2}{(1-k)\,\frac{t_{max}}{t_c}\,\overline{M}\,f[M]}\Bigg|_{\overline{P}}^{1}=\overline{t} \tag{13.9}$$

The analytical solution is applicable only in the case which the flow is choked thorough all the process. The solution is applicable to indirect connection. This happen when vacuum is applied outside the tube (a technique used in die casting and injection molding to improve quality by reducing porosity.). In case when the flow chokeless a numerical integration needed to be performed. In the literature, to create a direct function equation (13.4) is transformed into

$$\frac{d\overline{P}}{d\overline{t}}=\frac{k\left(1-\frac{t_{max}}{t_c}\,\overline{M}f[M]\overline{P}^{\frac{k-1}{2\,k}}\right)}{1-\overline{t}} \tag{13.10}$$

with the initial condition of

$$P(0)=1 \tag{13.11}$$

The analytical solution also can be approximated by a simpler equation as

$$\overline{P}=[1-t]^{\frac{t_{max}}{t_c}} \tag{13.12}$$

The results for numerical evaluation in the case when cylinder is initially at an atmospheric pressure and outside tube is also at atmospheric pressure are presented in Figure 13.2. In this case only some part of the flow is choked (the later part). The results of a choked case are presented in Figure (13.3) in which outside tube condition is in vacuum. These Figures 13.2 and 13.3 demonstrate the importance of the ratio of $\frac{t_{max}}{t_c}$. When $\frac{t_{max}}{t_c}>1$ the pressure increases significantly and verse versa.

Thus, the question remains, how the time ratio can be transferred to parameters that can the engineer can design in the system.

Denoting the area that creates the ratio $\frac{t_{max}}{t_c}=1$ as the critical area, A_c provides the needed tool. Thus the exit area, A can be expressed as

$$A=\frac{A}{A_c}\,A_c \tag{13.13}$$

The actual times ratio $\left.\frac{t_{max}}{t_c}\right|_{@A}$ can be expressed as

$$\left.\frac{t_{max}}{t_c}\right|_{@A}=\left.\frac{t_{max}}{t_c}\right|_{@A}\overbrace{\left.\frac{t_{max}}{t_c}\right|_{@A_c}}^{1} \tag{13.14}$$

According to equation (12.5) t_c is inversely proportional to area, $t_c \propto 1/A$. Thus, equation (13.14) the t_{max} is canceled and reduced into

$$\left.\frac{t_{max}}{t_c}\right|_{@A} = \frac{A}{A_c} \tag{13.15}$$

Parameters influencing the process are the area ratio, $\frac{A}{A_c}$, and the friction parameter, $\frac{4fL}{D}$. From other detailed calculations the author's thesis (later to be published at this site: www.potto.org). it was found that the influence of the parameter $\frac{4Flo}{D}$ on the pressure development in the cylinder is quite small. The influence is small on the residual air mass in the cylinder but larger on the Mach number, M_{exit}. The effects of the area ratio, $\frac{A}{A_c}$, are studied here since it is the dominant parameter.

It is important to point out the significance of the $\frac{t_{max}}{t_c}$. This parameter represents the ratio between the filling time and the evacuating time, the time which would be required to evacuate the cylinder for constant mass flow rate at the maximum Mach number when the gas temperature and pressure remain in their initial values. This parameter also represents the dimensionless area, $\frac{A}{A_c}$, according to the following equation

Figure 13.4 describes the pressure as a function of the dimensionless time for various values of $\frac{A}{A_c}$. The line that represents $\frac{A}{A_c} = 1$ is almost straight.

For large values of $\frac{A}{A_c}$ the pressure increases the volume flow rate of the air until a quasi steady state is reached. This quasi steady state is achieved when the volumetric air flow rate out is equal to the volume pushed by the piston. The pressure and the mass flow rate are maintained constant after this state is reached. The pressure in this quasi steady state is a function of $\frac{A}{A_c}$. For small values of $\frac{A}{A_c}$ there is no steady state stage. When $\frac{A}{A_c}$ is greater than one the pressure is concave upward and when $\frac{A}{A_c}$ is less than one the pressure is concave downward as shown in Figures (13.4), which was obtained by an integration of equation (13.9).

13.1.2 Examples

Example 13.1:
Calculate the minimum required vent area for die casting process when the die volume is $0.001[m^3]$ *and* $\frac{4fL}{D} = 20$. *The required solidification time,* $t_{max} = 0.03[sec]$.

SOLUTION

End Solution

13.1.3 Direct Connection

In the above analysis is applicable to indirect connection. It should be noted that critical area, A_c, is not function of the time. The direct connection posts more mathematical difficulty because the critical area is not constant and time dependent.

To continue

13.2 Non–Linear Functions Effects

In the above analysis a major assumption was dealing with the force or the function dictating the piston position or movement. The linear function represents for some manufacturing processes such as die casting. However, many other processes such as combustion engine are better represented by a periodic operation close to the sine function. The non-constant piston velocity leads to introduction of the energy transfer from the piston the fluid. This transfer, from dimensional analysis point of view, occurs at very low Mach number because the high speed of sound. Hence, at this range the main mechanism is via viscosity and there is no shock wave generating significant heat (transfer). When the piston is compressing gas, the speed of sound is significantly lower and therefor the Mach numbers are larger. Hence, the main mechanism energy transfer occur due to the shock dynamics.

Chapter 6 presents the reasons for shock when the piston velocity is larger than the gas velocity. It has to be repeated and emphasize two situations that causes shocks in the periodic cycle. The first one is the partially open value and second one is the partially close value. Consider the first case, when the piston acceleration is positive and piston is compressing the gas. That is, the piston velocity is larger than the displaced gas. In that case, the piston creates a new shock shown in a moccasin color exhibits Figure 13.5. This shock is moving faster than the piston itself. Hence, a quasi state condition is obtained for every state. The maximum amount of energy is limited by Bar–Meir's Maximum energy/temperature theory. The actual energy was suggested by Menikoff. According to Menikoff, two shocks (outgoing shock with the first successive shock) are sufficient to account for the actual energy transfer (see for detailed explanation in page 182). Hence, every element of the stage of the 1/4 of the cycle has to be accounted for. There is not known analytical method in carry this integration. The investigation carried at Potto Project shows that this effect increase the temperature significantly.

Very similar phenomenon can be observed on the other side of the cycle. Initially when the piston is expanded the gas velocity is smaller than the piston velocity. Later the gas expansion increases the gas velocity. At three quarters part of the cycle the piston velocity becomes small than the gas velocity. Similar situation to the close valve situation occurs. The piston produces shock at every element thereafter (till the end of the cycle). Again the Menikoff's suggestion to estimate can be applied for this case as well. These two zones are the main mechanism for energy transfer to the gas. The algorithm was build and test in Potto Project GDC and the publishing will appear shortly.

13.3 Summary

The analysis indicates there is a critical vent area below which the ventilation is poor and above which the resistance to air flow is minimal. This critical area depends on the geometry and the filling time. The critical area also provides a mean to "combine" the

actual vent area with the vent resistance for numerical simulations of the cavity filling, taking into account the compressibility of the gas flow.

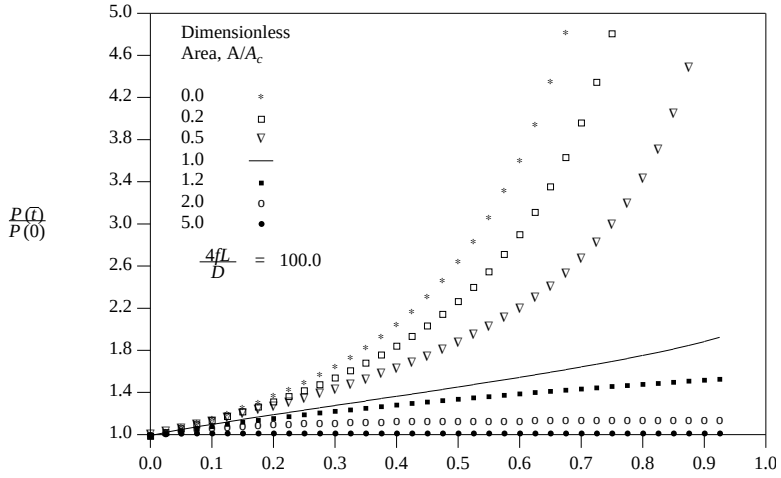


Figure a

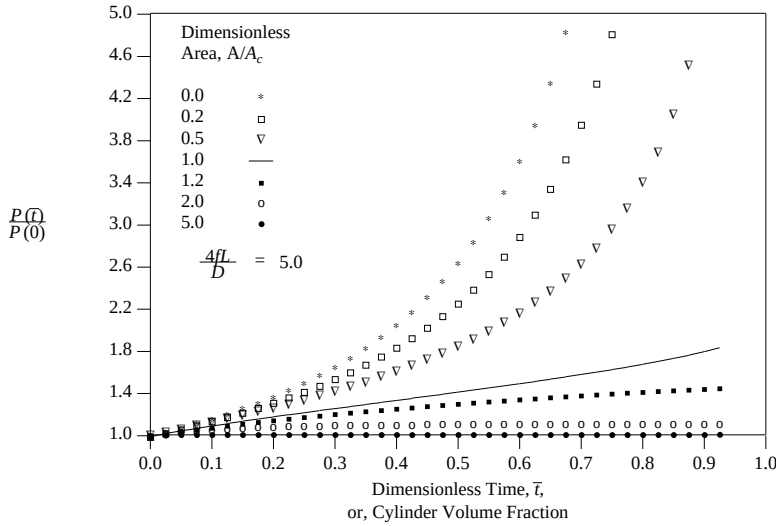


Figure b

Fig. -13.2. The pressure ratio as a function of the dimensionless time for chokeless condition.

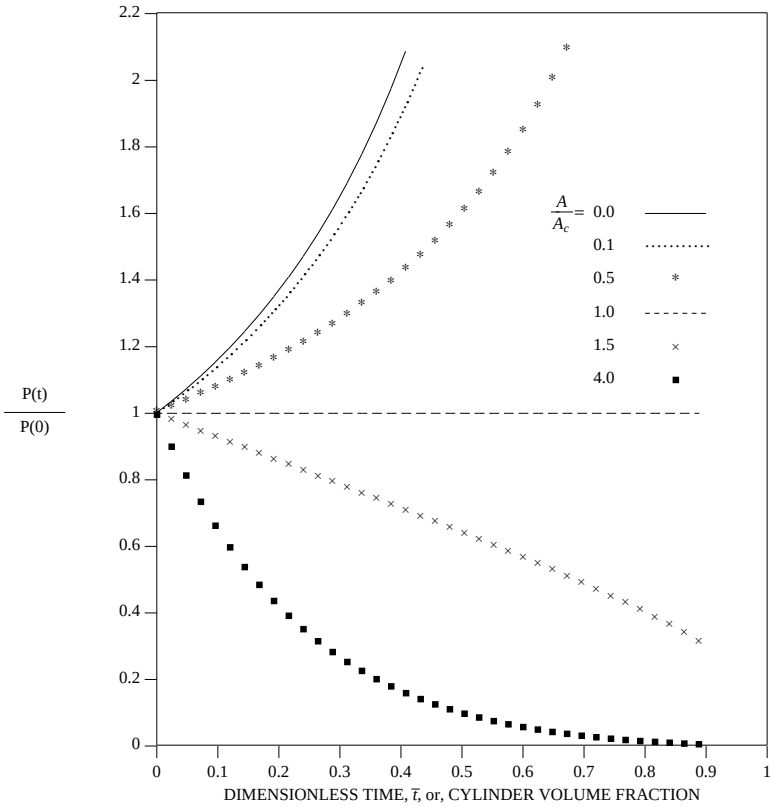


Fig. -13.3. The pressure ratio as a function of the dimensionless time for choked condition.

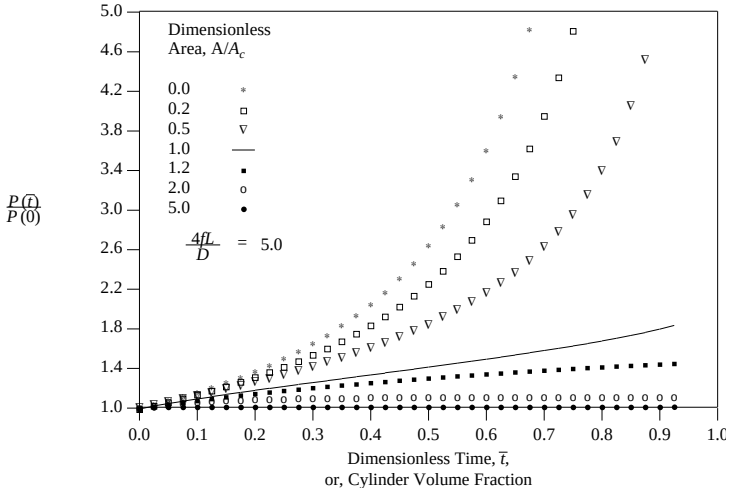


Fig. -13.4. The pressure ratio as a function of the dimensionless time.

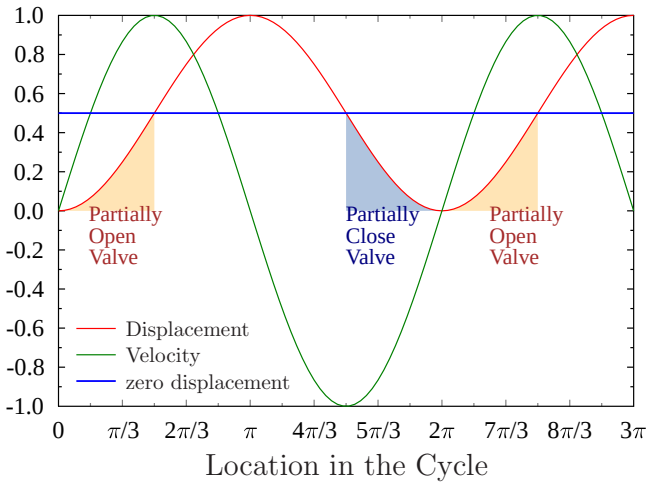


Fig. -13.5. The displacement of a piston and velocity under assumption of constant angular velocity. Every cycle has to regions one with partially open valve and one partially close valve.

CHAPTER 14

Oblique Shock

14.1 Preface to Oblique Shock

In Chapter (6), a discussion on a normal shock was presented. A normal shock is a special type of shock wave. The other type of shock wave is the oblique shock. In the literature oblique shock, normal shock, and Prandtl–Meyer function are presented as three separate and different issues. However, one can view all these cases as three different regions of a flow over a plate with a deflection section. Clearly, variation of the deflection angle from a zero ($\delta = 0$) to a positive value results in oblique shock. Further changing the deflection angle to a negative value results in expansion waves. The common representation is done by ignoring the boundaries of these models. However, this section attempts to show the boundaries and the limits or connections of these models[1].

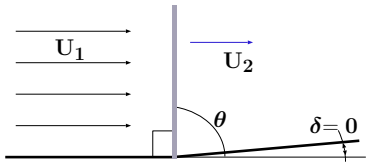


Fig. -14.1. A view of a straight normal shock as a limited case for oblique shock.

[1] In this chapter, even the whole book, a very limited discussion about reflection shocks and collisions of weak shock, Von Neumann paradox, triple shock intersection, etc are presented. The author believes that these issues are not relevant to most engineering students and practices. Furthermore, these issues should not be introduced in introductory textbook of compressible flow. Those who would like to obtain more information, should refer to J.B. Keller, "Rays, waves and asymptotic," Bull. Am. Math. Soc. 84, 727 (1978), and E.G. Tabak and R.R. Rosales, "Focusing of weak shock waves and the Von Neuman paradox of oblique shock reflection," Phys. Fluids 6, 1874 (1994).

14.2 Introduction

14.2.1 Introduction to Oblique Shock

A normal shock occurs when there is a disturbance downstream which imposes a boundary condition on the flow in which the fluid/gas can react only by a sharp change in the flow direction. As it may be recalled, normal shock occurs when a wall is straight/flat ($\delta = 0$) as shown in Figure (14.1) which occurs when somewhere downstream a disturbance[2] appears. When the deflection angle is increased, the gas flow must match the boundary conditions. This matching can occur only when there is a discontinuity in the flow field. Thus, the direction of the flow is changed by a shock wave with an angle to the flow. This shock is commonly referred to as the oblique shock. Alternatively, as discussed in Chapter 3 the flow behaves as it does in a hyperbolic field. In such a case, the flow field is governed by a hyperbolic equation which deals with the case when information (like boundary conditions) reaches from downstream only if they are within the range of influence. For information such as the disturbance (boundary condition) to reach deep into the flow from the side requires time. During this time, the flow moves downstream and creates an angle.

14.2.2 Introduction to Prandtl–Meyer Function

Decreasing the deflection angle results in the same effects as before. The boundary conditions must match the geometry. Yet, for a negative deflection angle (in this section's notation), the flow must be continuous. The analysis shows that the flow velocity must increase to achieve this requirement. This velocity increase is referred to as the expansion wave. As it will be shown in the next chapter, as opposed to oblique shock analysis, the increase in the upstream Mach number determines the downstream Mach number and the "negative" deflection angle.

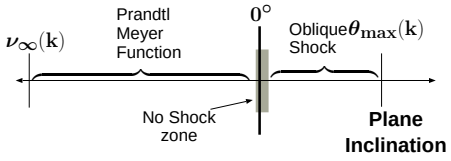


Fig. -14.2. The regions where oblique shock or Prandtl–Meyer function exist. Notice that both have a maximum point and a "no solution" zone, which is around zero. However, Prandtl-Meyer function approaches closer to a zero deflection angle.

It has to be pointed out that both the oblique shock and the Prandtl–Meyer function have a maximum point for $M_1 \to \infty$. However, the maximum point for the Prandtl–Meyer function is much larger than the oblique shock by a factor of more than 2. What accounts for the larger maximum point is the effective turning (less entropy production) which will be explained in the next chapter (see Figure (14.2)).

[2] Zero velocity, pressure boundary conditions, and different inclination angle, are examples of forces that create shock. The zero velocity can be found in a jet flowing into a still medium of gas.

14.2.3 Introduction to Zero Inclination

What happens when the inclination angle is zero? Which model is correct to use? Can these two conflicting models, the oblique shock and the Prandtl–Meyer function, co-exist? Or perhaps a different model better describes the physics. In some books and in the famous NACA report 1135 it was assumed that Mach wave and oblique shock co–occur in the same zone. Previously (see Chapter 6), it was assumed that normal shock occurs at the same time. In this chapter, the stability issue will be examined in greater detail.

14.3 Oblique Shock

The shock occurs in reality in situations where the shock has three–dimensional effects. The three–dimensional effects of the shock make it appear as a curved plane. However, one–dimensional shock can be considered a representation for a chosen arbitrary accuracy with a specific small area. In such a case, the change of the orientation makes the shock considerations two–dimensional. Alternately, using an infinite (or a two–dimensional) object produces a two–dimensional shock. The two–dimensional effects occur when the flow is affected from the "side," i.e., a change in the flow direction[3]. An example of such case is creation of shock from the side.

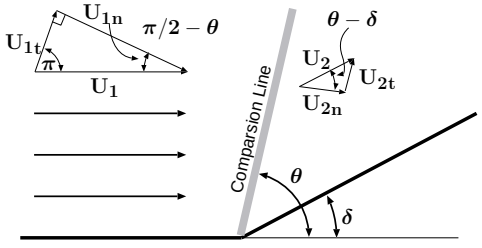


Fig. -14.3. A typical oblique shock schematic.

To match the boundary conditions, the flow turns after the shock to be parallel to the inclination angle. Figure (14.3) exhibits the schematic of the oblique shock. The deflection angle, δ, is the direction of the flow after the shock (parallel to the wall). The normal shock analysis dictates that after the shock, the flow is always subsonic. The total flow after the oblique shock can also be supersonic, which depends on the boundary layer and deflection angle.

The velocity has two components (with respect to the shock plane/surface). Only the oblique shock's normal component undergoes the "shock." The tangent component does not change because it does not "move" across the shock line. Hence, the mass balance reads

$$\rho_1 U_{1n} = \rho_2 U_{2n} \tag{14.1}$$

The momentum equation reads

$$P_1 + \rho_1 {U_{1n}}^2 = P_2 + \rho_2 {U_{2n}}^2 \tag{14.2}$$

[3]The author begs for forgiveness from those who view this description as offensive (There was an unpleasant email to the author accusing him of revolt against the holy of the holies.). If you do not like this description, please just ignore it. You can use the traditional explanation, you do not need the author's permission.

The momentum equation in the tangential direction is reduced to

$$U_{1t} = U_{2t} \tag{14.3}$$

The energy balance in coordinates moving with shock reads

$$C_p T_1 + \frac{{U_{1n}}^2}{2} = C_p T_2 + \frac{{U_{2n}}^2}{2} \tag{14.4}$$

Equations (14.1), (14.2), and (14.4) are the same as the equations for normal shock with the exception that the total velocity is replaced by the perpendicular components. Yet, the new relationship between the upstream Mach number, the deflection angle, δ, and the Mach angle, θ has to be solved. From the geometry it can be observed that

$$\tan\theta = \frac{U_{1n}}{U_{1t}} \tag{14.5}$$

and

$$\tan(\theta - \delta) = \frac{U_{2n}}{U_{2t}} \tag{14.6}$$

Unlike in the normal shock, here there are three possible pairs[4] of solutions to these equations. The first is referred to as the weak shock; the second is the strong shock; and the third is an impossible solution (thermodynamically)[5]. Experiments and experience have shown that the common solution is the weak shock, in which the shock turns to a lesser extent[6].

$$\frac{\tan\theta}{\tan(\theta - \delta)} = \frac{U_{1n}}{U_{2n}} \tag{14.7}$$

The above velocity–geometry equations can also be expressed in term of Mach number, as

$$\sin\theta = \frac{M_{1n}}{M_1} \tag{14.8}$$

and in the downstream side reads

$$\sin(\theta - \delta) = \frac{M_{2n}}{M_2} \tag{14.9}$$

Equation (14.8) alternatively also can be expressed as

$$\cos\theta = \frac{M_{1t}}{M_1} \tag{14.10}$$

[4]This issue is due to R. Menikoff, who raised the solution completeness issue.

[5]The solution requires solving the entropy conservation equation. The author is not aware of "simple" proof and a call to find a simple proof is needed.

[6]Actually this term is used from historical reasons. The lesser extent angle is the unstable angle and the weak angle is the middle solution. But because the literature referred to only two roots, the term lesser extent is used.

And equation (14.9) alternatively also can be expressed as

$$\cos\,(\theta-\delta)=\frac{M_{2t}}{M_2} \tag{14.11}$$

The total energy across a stationary oblique shock wave is constant, and it follows that the **total** speed of sound is constant across the (oblique) shock. It should be noted that although, $U_{1t}=U_{2t}$ the Mach number is $M_{1t}\neq M_{2t}$ because the temperatures on both sides of the shock are different, $T_1\neq T_2$.

As opposed to the normal shock, here angles (the second dimension) have to be determined. The solution from this set of four equations, (14.8) through (14.11), is a function of four unknowns of M_1, M_2, θ, and δ. Rearranging this set utilizing geometrical identities such as $\sin\alpha=2\sin\alpha\cos\alpha$ results in

Angle Relationship

$$\tan\delta=2\,\cot\theta\left[\frac{{M_1}^2\,\sin^2\theta-1}{{M_1}^2\,(k+\cos\,2\theta)+2}\right] \tag{14.12}$$

The relationship between the properties can be determined by substituting $M_1\sin\theta$ for of M_1 into the normal shock relationship, which results in

Pressure Ratio

$$\frac{P_2}{P_1}=\frac{2\,k\,{M_1}^2\sin^2\theta-(k-1)}{k+1} \tag{14.13}$$

The density and normal velocity ratio can be determined by the following equation

Density Ratio

$$\frac{\rho_2}{\rho_1}=\frac{U_{1n}}{U_{2n}}=\frac{(k+1){M_1}^2\sin^2\theta}{(k-1){M_1}^2\sin^2\theta+2} \tag{14.14}$$

The temperature ratio is expressed as

Temperature Ratio

$$\frac{T_2}{T_1}=\frac{2\,k\,{M_1}^2\sin^2\theta-(k-1)\left[(k-1){M_1}^2+2\right]}{(k+1)^2\,M_1} \tag{14.15}$$

Prandtl's relation for oblique shock is

$$U_{n_1}U_{n_2}=c^2-\frac{k-1}{k+1}{U_t}^2 \tag{14.16}$$

The Rankine–Hugoniot relations are the same as the relationship for the normal shock

$$\frac{P_2-P_1}{\rho_2-\rho_1}=k\frac{P_2-P_1}{\rho_2-\rho_1} \tag{14.17}$$

14.4 Solution of Mach Angle

Oblique shock, if orientated to a coordinate perpendicular and parallel shock plane is like a normal shock. Thus, the relationship between the properties can be determined by using the normal components or by utilizing the normal shock table developed earlier. One has to be careful to use the normal components of the Mach numbers. The stagnation temperature contains the total velocity.

Again, the normal shock is a one–dimensional problem, thus, only one parameter is required (to solve the problem). Oblique shock is a two–dimensional problem and two properties must be provided so a solution can be found. Probably, the most useful properties are upstream Mach number, M_1 and the deflection angle, which create a somewhat complicated mathematical procedure, and this will be discussed later. Other combinations of properties provide a relatively simple mathematical treatment, and the solutions of selected pairs and selected relationships will be presented.

14.4.1 Upstream Mach Number, M_1, and Deflection Angle, δ

Again, this set of parameters is, perhaps, the most common and natural to examine. Thompson (1950) has shown that the relationship of the shock angle is obtained from the following cubic equation:

Governing Angle Equation

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0 \tag{14.18}$$

where

$$x = \sin^2 \theta \tag{14.19}$$

and

$$a_1 = -\frac{{M_1}^2 + 2}{{M_1}^2} - k \sin^2 \delta \tag{14.20}$$

$$a_2 = -\frac{2{M_1}^2 + 1}{{M_1}^4} + \left[\frac{(k+1)^2}{4} + \frac{k-1}{{M_1}^2} \right] \sin^2 \delta \tag{14.21}$$

$$a_3 = -\frac{\cos^2 \delta}{{M_1}^4} \tag{14.22}$$

Equation (14.18) requires that x has to be a real and positive number to obtain a real deflection angle[7]. Clearly, $\sin\theta$ must be positive, and the negative sign refers to the mirror image of the solution. Thus, the negative root of $\sin\theta$ must be disregarded

[7] This point was pointed out by R. Menikoff. He also suggested that θ is bounded by $\sin^{-1} 1/M_1$ and 1.

The solution of a cubic equation such as (14.18) provides three roots[8]. These roots can be expressed as

First Root

$$x_1 = -\frac{1}{3}a_1 + (S+T) \tag{14.23}$$

Second Root

$$x_2 = -\frac{1}{3}a_1 - \frac{1}{2}(S+T) + \frac{1}{2}i\sqrt{3}(S-T) \tag{14.24}$$

and

Third Root

$$x_3 = -\frac{1}{3}a_1 - \frac{1}{2}(S+T) - \frac{1}{2}i\sqrt{3}(S-T) \tag{14.25}$$

Where

$$S = \sqrt[3]{R+\sqrt{D}}, \tag{14.26}$$

$$T = \sqrt[3]{R-\sqrt{D}} \tag{14.27}$$

and where the definition of the D is

$$D = Q^3 + R^2 \tag{14.28}$$

and where the definitions of Q and R are

$$Q = \frac{3a_2 - {a_1}^2}{9} \tag{14.29}$$

and

$$R = \frac{9a_1a_2 - 27a_3 - 2{a_1}^3}{54} \tag{14.30}$$

Only three roots can exist for the Mach angle, θ. From a mathematical point of view, if $D > 0$, one root is real and two roots are complex. For the case $D = 0$, all the roots are real and at least two are identical. In the last case where $D < 0$, all the roots are real and unequal.

The physical meaning of the above analysis demonstrates that in the range where $D > 0$ no solution can exist because no imaginary solution can exist[9]. $D > 0$ occurs

[8] The highest power of the equation (only with integer numbers) is the number of the roots. For example, in a quadratic equation there are two roots.

[9] A call for suggestions, to explain about complex numbers and imaginary numbers should be included. Maybe insert an example where imaginary solution results in no physical solution.

when no shock angle can be found, so that the shock normal component is reduced to subsonic and yet parallel to the inclination angle.

Furthermore, only in some cases when $D = 0$ does the solution have a physical meaning. Hence, the solution in the case of $D = 0$ has to be examined in the light of other issues to determine the validity of the solution.

When $D < 0$, the three unique roots are reduced to two roots at least for the steady state because thermodynamics dictates[10] that. Physically, it can be shown that the first solution(14.23), referred sometimes as a thermodynamically unstable root, which is also related to a decrease in entropy, is "unrealistic." Therefore, the first solution does not occur in reality, at least, in steady–state situations. This root has only a mathematical meaning for steady–state analysis[11].

These two roots represent two different situations. First, for the second root, the shock wave keeps the flow almost all the time as a supersonic flow and it is referred to as the weak solution (there is a small section that the flow is subsonic). Second, the third root always turns the flow into subsonic and it is referred to as the strong solution. It should be noted that this case is where entropy increases in the largest amount.

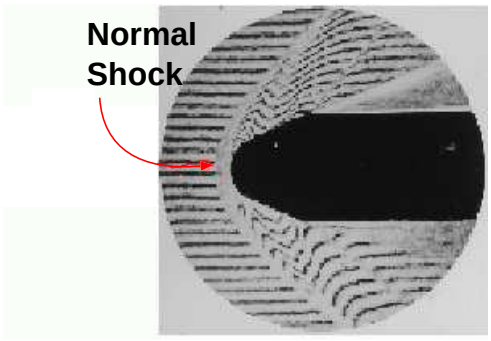


Fig. -14.4. Flow around spherically blunted 30° cone-cylinder with Mach number 2.0. It can be noticed that the normal shock, the strong shock, and the weak shock coexist.

In summary, if an imaginary hand moves the shock angle starting from the deflection angle and reaching the first angle that satisfies the boundary condition, this situation is unstable and the shock angle will jump to the second angle (root). If an additional "push" is given, for example, by additional boundary conditions, the shock angle will jump to the third root[12]. These two angles of the strong and weak shock are stable for a two–dimensional wedge (see the appendix of this chapter for a limited discussion on the

[10] This situation is somewhat similar to a cubical body rotation. The cubical body has three symmetrical axes which the body can rotate around. However, the body will freely rotate only around two axes with small and large moments of inertia. The body rotation is unstable around the middle axes. The reader can simply try it.

[11] There is no experimental or analytical evidence, that the author has found, showing that it is totally impossible. The "unstable" terms can be thermodynamically stable in unsteady case. Though, those who are dealing with rapid transient situations should be aware that this angle of oblique shock can exist. There is no theoretical evidence that showing that in strong unsteady state this angle is unstable. The shock will initially for a very brief time transient in it and will jump from this angle to the thermodynamically stable angles.

[12] See the discussion on the stability. There are those who view this question not as a stability equation but rather as under what conditions a strong or a weak shock will prevail.

stability[13]).

14.4.2 When No Oblique Shock Exist or the case of $D > 0$

Large deflection angle for given, M_1

The first range is when the deflection angle reaches above the maximum point. For a given upstream Mach number, M_1, a change in the inclination angle requires a larger energy to change the flow direction. Once, the inclination angle reaches the "maximum potential energy," a change in the flow direction is no longer possible. As the alternative view, the fluid "sees" the disturbance (in this case, the wedge) in front of it and hence the normal shock occurs. Only when the fluid is away from the object (smaller angle) liquid "sees" the object in a different inclination angle. This different inclination angle is sometimes referred to as an imaginary angle.

The Simple Calculation Procedure

For example, in Figure (14.4) and (14.5), the imaginary angle is shown. The flow is far away from the object and does not "see' the object. For example, for, $M_1 \longrightarrow \infty$ the maximum deflection angle is calculated when $D = Q^3 + R^2 = 0$. This can be done by evaluating the terms a_1, a_2, and a_3 for $M_1 = \infty$.

$$a_1 = -1 - k\sin^2\delta$$
$$a_2 = \frac{(k+1)^2\sin^2\delta}{4}$$
$$a_3 = 0$$

With these values the coefficients R and Q are

$$R = \frac{-9(1+k\sin^2\delta)\left(\frac{(k+1)^2\sin^2\delta}{4}\right) - (2)(-)(1+k\sin^2\delta)^2}{54}$$

and

$$Q = \frac{(1+k\sin^2\delta)^2}{9}$$

Solving equation (14.28) after substituting these values of Q and R provides series of roots from which only one root is possible. This root, in the case $k = 1.4$, is just above $\delta_{max} \sim \frac{\pi}{4}$ (note that the maximum is also a function of the heat ratio, k).

While the above procedure provides the general solution for the three roots, there is simplified transformation that provides solution for the strong and and weak solution. It must be noted that in doing this transformation, the first solution is "lost" supposedly because it is "negative." In reality the first solution is not negative but rather some value between zero and the weak angle. Several researchers[14] suggested

[13]This material is extra and not recommended for standard undergraduate students.

[14]A whole discussion on the history of this can be found in "Open content approach to academic writing" on `http://www.potto.org/obliqueArticle.phpattheendofthebook`.

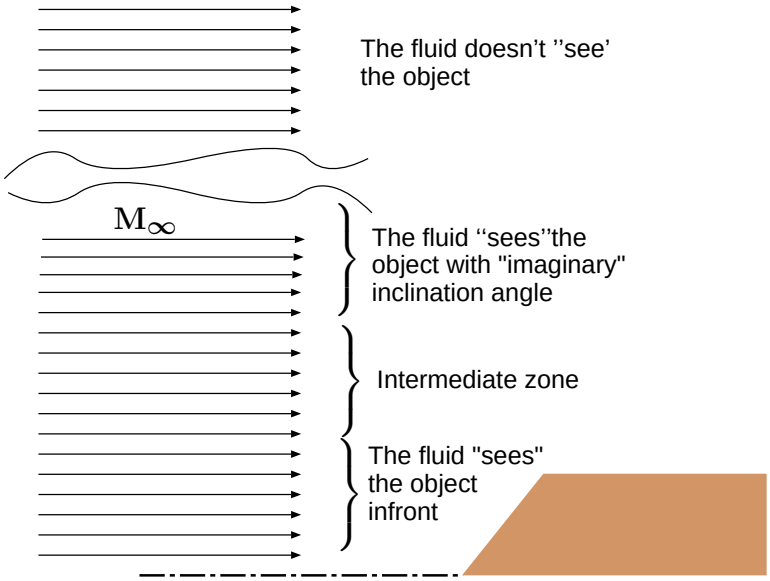


Fig. -14.5. The view of a large inclination angle from different points in the fluid field.

that instead Thompson's equation should be expressed by equation (14.18) by $\tan\theta$ and is transformed into

$$\left(1+\frac{k-1}{2}{M_1}^2\right)\tan\delta\tan^3\theta-\left({M_1}^2-1\right)\tan^2\theta+\left(1+\frac{k+1}{2}\right)\tan\delta\tan\theta+1=0 \tag{14.31}$$

The solution to this equation (14.31) for the weak angle is

Weak Angle Solution

$$\theta_{weak}=\tan^{-1}\left(\frac{{M_1}^2-1+2\,f_1(M_1,\delta)\,\cos\left(\frac{4\,\pi+\cos^{-1}(f_2(M_1,\delta))}{3}\right)}{3\left(1+\frac{k-1}{2}\,{M_1}^2\right)\tan\delta}\right) \tag{14.32}$$

Strong Angle Solution

$$\theta_{strong} = \tan^{-1} \frac{{M_1}^2 - 1 + 2 f_1(M_1, \delta) \cos\left(\frac{\cos^{-1}(f_2(M_1, \delta))}{3}\right)}{3\left(1 + \frac{k-1}{2} {M_1}^2\right) \tan\delta} \tag{14.33}$$

where these additional functions are

$$f_1(M_1, \delta) = \sqrt{\left({M_1}^2 - 1\right)^2 - 3\left(1 + \frac{k-1}{2} {M_1}^2\right)\left(1 + \frac{k+1}{2} {M_1}^2\right) \tan^2\delta} \tag{14.34}$$

and

$$f_2(M_1, \delta) = \frac{\left({M_1}^2 - 1\right)^3 - 9\left(1 + \frac{k-1}{2} {M_1}^2\right)\left(1 + \frac{k-1}{2} {M_1}^2 + \frac{k+1}{4} {M_1}^4\right) \tan^2\delta}{f_1(M_1, \delta)^3} \tag{14.35}$$

Figure (14.6) exhibits typical results for oblique shock for two deflection angle of 5 and 25 degree. Generally, the strong shock is reduced as the increase of the Mach number while the weak shock is increase. The impossible shock for unsteady state is almost linear function of the upstream Mach number and almost not affected by the deflection angle.

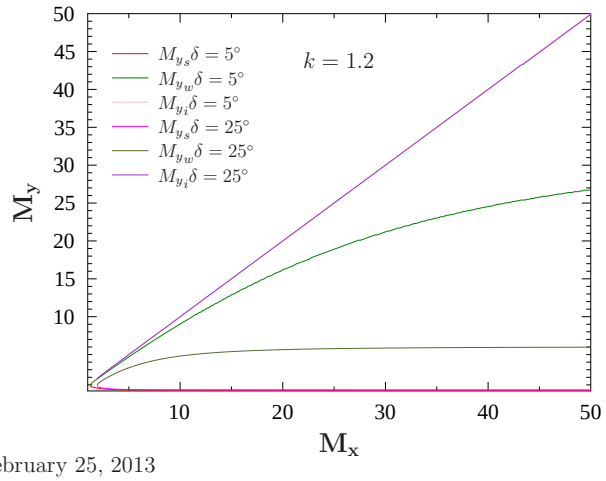


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Fig. -14.6. The three different Mach numbers after the oblique shock for two deflection angles of $5°$ *and* $25°$.

The Procedure for Calculating The Maximum Deflection Point

The maximum angle is obtained when $D = 0$. When the right terms defined in (14.20)-(14.21), (14.29), and (14.30) are substituted into this equation and utilizing the trigonometrical identity $\sin^2\delta + \cos^2\delta = 1$ and other trigonometrical identities results in Maximum Deflection Mach Number's equation in which is

$$M_1{}^2\,(k+1)\,(M_{1n}{}^2+1) = 2(kM_{1n}{}^4 + 2M_{1n}{}^2 - 1) \tag{14.36}$$

This equation and its twin equation can be obtained by an alternative procedure proposed by someone[15] who suggested another way to approach this issue. It can be noticed that in equation (14.12), the deflection angle is a function of the Mach angle and the upstream Mach number, M_1. Thus, one can conclude that the maximum Mach angle is only a function of the upstream Much number, M_1. This can be shown mathematically by the argument that differentiating equation (14.12) and equating the results to zero creates relationship between the Mach number, M_1 and the maximum Mach angle, θ. Since in that equation there appears only the heat ratio k, and Mach number, M_1, θ_{max} is a function of only these parameters. The differentiation of the equation (14.12) yields

$$\frac{d\tan\delta}{d\theta} = \frac{kM_1{}^4\sin^4\theta + \left(2 - \frac{(k+1)}{2}M_1{}^2\right)M_1{}^2\sin^2\theta - \left(1 + \frac{(k+1)}{2}M_1{}^2\right)}{kM_1{}^4\sin^4\theta - \left[(k-1) + \frac{(k+1)^2M_1{}^2}{4}\right]M_1{}^2\sin^2\theta - 1} \tag{14.37}$$

Because $\tan$ is a monotonous function, the maximum appears when θ has its maximum. The numerator of equation (14.37) is zero at different values of the denominator. Thus, it is sufficient to equate the numerator to zero to obtain the maximum. The nominator produces a quadratic equation for $\sin^2\theta$ and only the positive value for $\sin^2\theta$ is applied here. Thus, the $\sin^2\theta$ is

$$\sin^2\theta_{max} = \frac{-1 + \frac{k+1}{4}M_1{}^2 + \sqrt{(k+1)\left[1 + \frac{k-1}{2}M_1{}^2 + \left(\frac{k+1}{2}M_1\right)^4\right]}}{kM_1{}^2} \tag{14.38}$$

Equation (14.38) should be referred to as the maximum's equation. It should be noted that both the Maximum Mach Deflection equation and the maximum's equation lead to the same conclusion that the maximum M_{1n} is only a function of upstream the Mach number and the heat ratio k. It can be noticed that the Maximum Deflection Mach Number's equation is also a quadratic equation for $M_{1n}{}^2$. Once M_{1n} is found, then the Mach angle can be easily calculated by equation (14.8). To compare these two equations the simple case of Maximum for an infinite Mach number is examined. It must be pointed out that similar procedures can also be proposed (even though it

[15] At first, it was seen as C. J.Chapman, English mathematician to be the creator but later an earlier version by several months was proposed by Bernard Grossman. At this stage, it is not clear who was the first to propose it.

does not appear in the literature). Instead, taking the derivative with respect to θ, a derivative can be taken with respect to M_1. Thus,

$$\frac{d\tan\delta}{dM_1} = 0 \tag{14.39}$$

and then solving equation (14.39) provides a solution for M_{max}.

A simplified case of the Maximum Deflection Mach Number's equation for large Mach number becomes

$$M_{1n} = \sqrt{\frac{k+1}{2k}} M_1 \quad \text{for} \quad M_1 >> 1 \tag{14.40}$$

Hence, for large Mach numbers, the Mach angle is $\sin\theta = \sqrt{\frac{k+1}{2k}}$ (for k=1.4), which makes $\theta = 1.18$ or $\theta = 67.79°$.

With the value of θ utilizing equation (14.12), the maximum deflection angle can be computed. Note that this procedure does not require an approximation of M_{1n} to be made. The general solution of equation (14.36) is

Normal Shock Minikoff Solution

$$M_{1n} = \frac{\sqrt{\sqrt{(k+1)^2\ {M_1}^4 + 8\ (k^2-1)\ {M_1}^2 + 16\ (k+1)} + (k+1)\ {M_1}^2 - 4}}{2\sqrt{k}} \tag{14.41}$$

Note that Maximum Deflection Mach Number's equation can be extended to deal with more complicated equations of state (aside from the perfect gas model).

This typical example is for those who like mathematics.

Example 14.1:
Derive the perturbation of Maximum Deflection Mach Number's equation for the case of a very small upstream Mach number number of the form $M_1 = 1 + \epsilon$. Hint, Start with equation (14.36) *and neglect all the terms that are relatively small.*

SOLUTION

The solution can be done by substituting $(M_1 = 1 + \epsilon)$ into equation (14.36) and it results in

Normal Shock Small Values

$$M_{1n} = \sqrt{\frac{\sqrt{\epsilon(k)} + \epsilon^2 + 2\,\epsilon - 3 + k\epsilon^2 + 2\,k\epsilon + k}{4k}} \tag{14.42}$$

where the epsilon function is

$$\begin{aligned}\epsilon(k) =&(k^2+2k+1)\,\epsilon^4+(4\,k^2+8\,k+4)\,\epsilon^3+\\&(14\,k^2+12\,k-2)\,\epsilon^2+(20\,k^2+8\,k-12)\,\epsilon+9\,(k+1)^2\end{aligned}\tag{14.43}$$

Now neglecting all the terms with ϵ results for the epsilon function in

$$\epsilon(k)\sim 9\,(k+1)^2\tag{14.44}$$

And the total operation results in

$$M_{1n}=\sqrt{\frac{3\,(k+1)-3+k}{4k}}=1\tag{14.45}$$

Interesting to point out that as a consequence of this assumption the maximum shock angle, θ is a normal shock. However, taking the second term results in different value. Taking the second term in the explanation results in

$$M_{1n}=\sqrt{\frac{\sqrt{9\,(k+1)^2+(20\,k^2+8\,k-12)\,\epsilon}-3+k+2\,(1+k)\epsilon}{4k}}\tag{14.46}$$

Note this equation (14.46) produce an un realistic value and additional terms are required to obtained to produce a realistic value.

End Solution

The case of $D\geq 0$ or $0\geq\delta$

The second range in which $D>0$ is when $\delta<0$. Thus, first the transition line in which $D=0$ has to be determined. This can be achieved by the standard mathematical procedure of equating $D=0$. The analysis shows regardless of the value of the upstream Mach number $D=0$ when $\delta=0$. This can be partially demonstrated by evaluating the terms a_1, a_2, and a_3 for the specific value of M_1 as following

$$\begin{aligned}a_1&=\frac{{M_1}^2+2}{{M_1}^2}\\a_2&=-\frac{2{M_1}^2+1}{{M_1}^4}\\a_3&=-\frac{1}{{M_1}^4}\end{aligned}\tag{14.47}$$

With values presented in equations (14.47) for R and Q becoming

$$\begin{aligned}R&=\frac{9\left(\frac{{M_1}^2+2}{{M_1}^2}\right)\left(\frac{2{M_1}^2+1}{{M_1}^4}\right)+27\left(\frac{1}{{M_1}^4}\right)-2\left(\frac{{M_1}^2+2}{{M_1}^2}\right)^2}{54}\\&=\frac{9\left({M_1}^2+2\right)\left(2{M_1}^2+1\right)+27{M_1}^2-2\,{M_1}^2\left({M_1}^2+2\right)^2}{54\,{M_1}^6}\end{aligned}\tag{14.48}$$

and

$$Q = \frac{3\left(\frac{2{M_1}^2+1}{{M_1}^4}\right) - \left(\frac{{M_1}^2+2}{{M_1}^2}\right)^3}{9} \tag{14.49}$$

Substituting the values of Q and R equations (14.48) (14.49) into equation (14.28) provides the equation to be solved for δ.

$$\left[\frac{3\left(\frac{2{M_1}^2+1}{{M_1}^4}\right) - \left(\frac{{M_1}^2+2}{{M_1}^2}\right)^3}{9}\right]^3 + \left[\frac{9\left({M_1}^2+2\right)\left(2{M_1}^2+1\right) + 27{M_1}^2 - 2{M_1}^2\left({M_1}^2+2\right)^2}{54{M_1}^6}\right]^2 = 0 \tag{14.50}$$

The author is not aware of any analytical demonstration in the literature which shows that the solution is identical to zero for $\delta = 0$[16]. Nevertheless, this identity can be demonstrated by checking several points for example, $M_1 = 1., 2.0, \infty$. Table (14.7) is provided for the following demonstration. Substitution of all the above values into (14.28) results in $D = 0$.

Utilizing the symmetry and antisymmetry of the qualities of the $\cos$ and $\sin$ for $\delta < 0$ demonstrates that $D > 0$ regardless of Mach number. Hence, the physical interpretation of this fact is that either no shock exists and the flow is without any discontinuity or that a normal shock exists[17]. Note that, in the previous case, with a positive large deflection angle, there was a transition from one kind of discontinuity to another.

In the range where $\delta \leq 0$, the question is whether it is possible for an oblique shock to exist? The answer according to this analysis and stability analysis is no. And according to this analysis, no Mach wave can be generated from the wall with **zero deflection**. In other words, the wall does not emit any signal to the flow (assuming zero viscosity), which contradicts the common approach. Nevertheless, in the literature, there are several papers

M_1 \ coefficients	a_1	a_2	a_3
1.0	-3	-1	$-\frac{3}{2}$
2.0	3	0	$\frac{9}{16}$
∞	-1	0	$-\frac{1}{16}$

Fig. -14.7. The various coefficients of three different Mach numbers to demonstrate that D is zero

[16] A mathematical challenge for those who like to work it out.

[17] There are several papers that attempt to prove this point in the past. Once this analytical solution was published, this proof became trivial. But for non ideal gas (real gas) this solution is only an indication.

suggesting zero strength Mach wave; others suggest a singular point[18]. The question of singular point or zero Mach wave strength are only of mathematical interest.

Suppose that there is a Mach wave at the wall at zero inclination (see Figure (14.8)). Obviously, another Mach wave occurs after a small distance. But because the velocity after a Mach wave (even for an extremely weak shock wave) is reduced, thus, the Mach angle will be larger ($\mu_2 > \mu_1$). If the situation keeps on occurring over a finite distance, there will be a point where the Mach number will be 1 and a normal shock will occur, according the common explanation. However, the reality is that no continuous Mach wave can occur because of the viscosity (boundary layer).

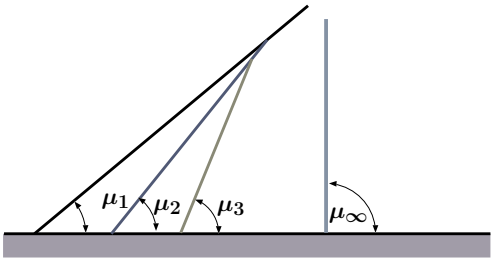


Fig. -14.8. The Mach waves that are supposed to be generated at zero inclination.

In reality, there are imperfections in the wall and in the flow and there is the question of boundary layer. It is well known, in the engineering world, that there is no such thing as a perfect wall. The imperfections of the wall can be, for simplicity's sake, assumed to be as a sinusoidal shape. For such a wall the zero inclination changes from small positive value to a negative value. If the Mach number is large enough and the wall is rough enough, there will be points where a weak[19] weak will be created. On the other hand, the boundary layer covers or smooths out the bumps. With these conflicting mechanisms, both will not allow a situation of zero inclination with emission of Mach wave. At the very extreme case, only in several points (depending on the bumps) at the leading edge can a very weak shock occur. Therefore, for the purpose of an introductory class, no Mach wave at zero inclination should be assumed.

Furthermore, if it was assumed that no boundary layer exists and the wall is perfect, any deviations from the zero inclination angle creates a jump from a positive angle (Mach wave) to a negative angle (expansion wave). This theoretical jump occurs because in a Mach wave the velocity decreases while in the expansion wave the velocity increases. Furthermore, the increase and the decrease depend on the upstream Mach number but in different directions. This jump has to be in reality either smoothed out or has a physical meaning of jump (for example, detach normal shock). The analysis started by looking at a normal shock which occurs when there is a zero inclination. After analysis of the oblique shock, the same conclusion must be reached, i.e. that the normal shock can occur at zero inclination. The analysis of the oblique shock suggests that the inclination angle is not the source (boundary condition) that creates the shock. There must be another boundary condition(s) that causes the normal shock. In the light

[18] See for example, paper by Rosles, Tabak, "Caustics of weak shock waves," 206 Phys. Fluids 10 (1) , January 1998.

[19] It is not a mistake, there are two "weaks." These words mean two different things. The first "weak" means more of compression "line" while the other means the weak shock.

of this discussion, at least for a simple engineering analysis, the zone in the proximity of zero inclination (small positive and negative inclination angle) should be viewed as a zone without any change unless the boundary conditions cause a normal shock.

Nevertheless, emission of Mach wave can occur in other situations. The approximation of weak weak wave with nonzero strength has engineering applicability in a very limited cases, especially in acoustic engineering, but for most cases it should be ignored.

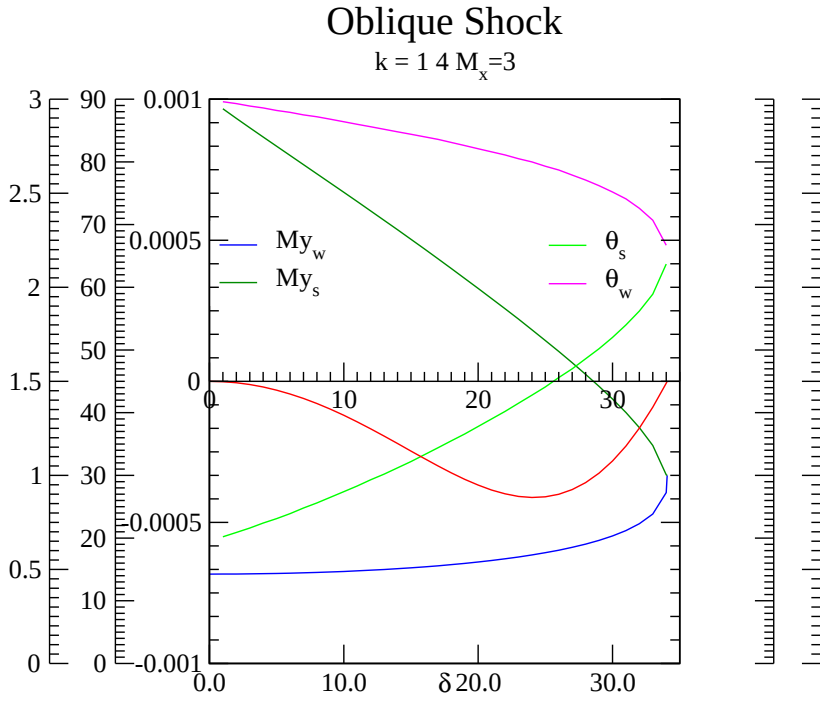


Fig. -14.9. The D, shock angle, and M_y for $M_1 = 3$.

14.4.3 Upstream Mach Number, M_1, and Shock Angle, θ

The solution for upstream Mach number, M_1, and shock angle, θ, are far much simpler and a unique solution exists. The deflection angle can be expressed as a function of

these variables as

δ For θ and M_1

$$\cot\delta = \tan(\theta)\left[\frac{(k+1)\,{M_1}^2}{2\,({M_1}^2\sin^2\theta - 1)} - 1\right] \tag{14.51}$$

or

$$\tan\delta = \frac{2\cot\theta({M_1}^2\sin^2\theta - 1)}{2 + {M_1}^2(k + 1 - 2\sin^2\theta)} \tag{14.52}$$

The pressure ratio can be expressed as

Pressure Ratio

$$\frac{P_2}{P_1} = \frac{2\,k\,{M_1}^2\sin^2\theta - (k-1)}{k+1} \tag{14.53}$$

The density ratio can be expressed as

Density Ratio

$$\frac{\rho_2}{\rho_1} = \frac{U_{1n}}{U_{2n}} = \frac{(k+1)\,{M_1}^2\,\sin^2\theta}{(k-1)\,{M_1}^2\,\sin^2\theta + 2} \tag{14.54}$$

The temperature ratio expressed as

Temperature Ratio

$$\frac{T_2}{T_1} = \frac{{c_2}^2}{{c_1}^2} = \frac{\left(2\,k\,{M_1}^2\sin^2\theta - (k-1)\right)\left((k-1){M_1}^2\sin^2\theta + 2\right)}{(k+1)\,{M_1}^2\,\sin^2\theta} \tag{14.55}$$

The Mach number after the shock is

Exit Mach Number

$${M_2}^2\sin(\theta - \delta) = \frac{(k-1){M_1}^2\sin^2\theta + 2}{2\,k\,{M_1}^2\sin^2\theta - (k-1)} \tag{14.56}$$

or explicitly

$${M_2}^2 = \frac{(k+1)^2{M_1}^4\sin^2\theta - 4\,({M_1}^2\sin^2\theta - 1)(k{M_1}^2\sin^2\theta + 1)}{\left(2\,k\,{M_1}^2\sin^2\theta - (k-1)\right)\left((k-1)\,{M_1}^2\sin^2\theta + 2\right)} \tag{14.57}$$

The ratio of the total pressure can be expressed as

Stagnation Pressure Ratio

$$\frac{P_{0_2}}{P_{0_1}} = \left[\frac{(k+1){M_1}^2\sin^2\theta}{(k-1){M_1}^2\sin^2\theta + 2}\right]^{\frac{k}{k-1}}\left[\frac{k+1}{2k{M_1}^2\sin^2\theta - (k-1)}\right]^{\frac{1}{k-1}} \tag{14.58}$$

Even though the solution for these variables, M_1 and θ, is unique, the possible range deflection angle, δ, is limited. Examining equation (14.51) shows that the shock angle, θ , has to be in the range of $\sin^{-1}(1/M_1) \geq \theta \geq (\pi/2)$ (see Figure 14.10). The range of given θ, upstream Mach number M_1, is limited between ∞ and $\sqrt{1/\sin^2\theta}$.

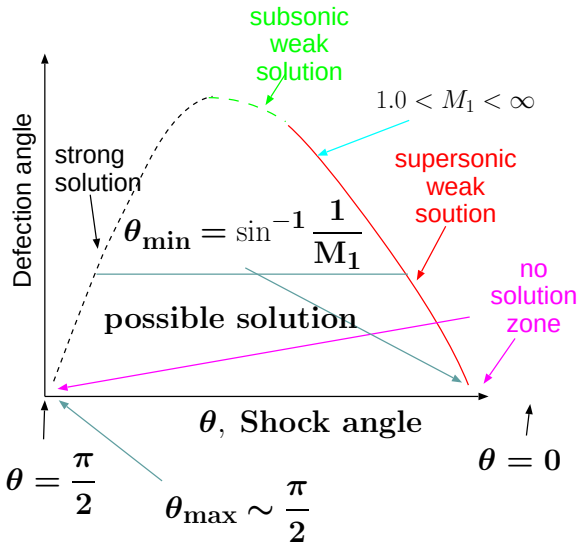


Fig. -14.10. The possible range of solutions for different parameters for given upstream Mach numbers.

14.4.4 Given Two Angles, δ and θ

It is sometimes useful to obtain a relationship where the two angles are known. The first upstream Mach number, M_1 is

Mach Number Angles Relationship

$$M_1{}^2 = \frac{2\left(\cot\theta + \tan\delta\right)}{\sin 2\theta - \left(\tan\delta\right)\left(k + \cos 2\theta\right)} \tag{14.59}$$

The reduced pressure difference is

$$\frac{2(P_2 - P_1)}{\rho U^2} = \frac{2\sin\theta\sin\delta}{\cos(\theta - \delta)} \tag{14.60}$$

The reduced density is

$$\frac{\rho_2 - \rho_1}{\rho_2} = \frac{\sin\delta}{\sin\theta\cos(\theta - \delta)} \tag{14.61}$$

For a large upstream Mach number M_1 and a small shock angle (yet not approaching zero), θ, the deflection angle, δ must also be small as well. Equation (14.51) can be simplified into

$$\theta \cong \frac{k+1}{2}\delta \tag{14.62}$$

The results are consistent with the initial assumption which shows that it was an appropriate assumption.

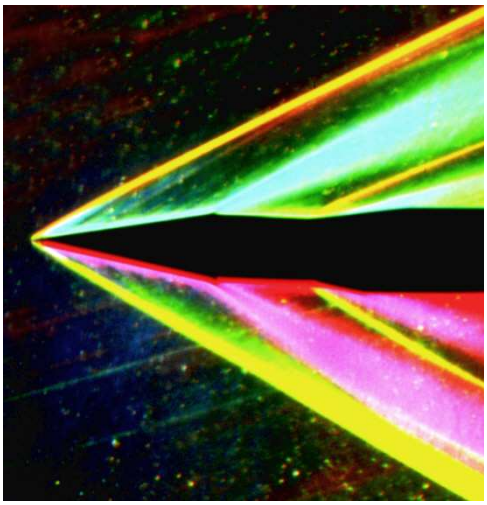


Fig. -14.11. Color-schlieren image of a two dimensional flow over a wedge. The total deflection angel (two sides) is 20° and upper and lower Mach angel are $\sim 28^\circ$ and $\sim 30^\circ$, respectively. The image show the end–effects as it has thick (not sharp transition) compare to shock over a cone. The image was taken by Dr. Gary Settles at Gas Dynamics laboratory, Penn State University.

14.4.5 Flow in a Semi–2D Shape

Example 14.2:
In Figure 14.11 exhibits wedge in a supersonic flow with unknown Mach number. Examination of the Figure reveals that it is in angle of attack. 1) Calculate the Mach number assuming that the lower and the upper Mach angles are identical and equal to $\sim 30^\circ$ each (no angle of attack). 2) Calculate the Mach number and angle of attack assuming that the pressure after the shock for the two oblique shocks is equal. 3) What kind are the shocks exhibits in the image? (strong, weak, unsteady) 4) (Open question) Is there possibility to estimate the air stagnation temperature from the information provided in the image. You can assume that specific heats, k is a monotonic increasing function of the temperature.

Solution

Part (1)

The Mach angle and deflection angle can be obtained from the Figure 14.11. With this data and either using equation (14.59) or potto-GDC results in

$\mathbf{M_1}$	$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.6810	2.3218	0	2.24	0	30	10	0.97172

The actual Mach number after the shock is then

$$M_2 = \frac{M_{2n}}{\sin(\theta - \delta)} = \frac{0.76617}{\sin(30 - 10)} = 0.839$$

The flow after the shock is subsonic flow.

Part (2)

For the lower part shock angle of $\sim 28^\circ$ the results are

$\mathbf{M_1}$	$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.9168	2.5754	0	2.437	0	28	10	0.96549

From the last table, it is clear that Mach number is between the two values of 2.9168 and 2.6810 and the pressure ratio is between 0.96549 and 0.97172. One of procedure to calculate the attack angle is such that pressure has to match by "guessing" the Mach number between the extreme values.

Part (3)

The shock must be weak shock because the shock angle is less than 60°.

End Solution

14.4.6 Flow in a Semi-2D Shape

The discussion so far was about the straight infinite long wedge[20] which is a "pure" 2–D configuration. Clearly, for any finite length of the wedge, the analysis needs to account for edge effects. The end of the wedge must have a different configuration (see Figure 14.12). Yet, the analysis for the middle section produces a close result to reality (because of symmetry). The section where the current analysis is close to reality can be estimated from a dimensional analysis for the required accuracy or by a numerical method. The dimensional analysis shows that only the doted area to be area

[20] Even finite wedge with limiting wall can be considered as an example for this discussion if the B.L. is neglected.

where current solution can be assumed as correct[21]. In spite of the small area were the current solution can be assumed, this solution is also act as a "reality check" to any numerical analysis. The analysis also provides additional value of the expected range. In Figure 14.11 shows that "shock angle" is not sharp. The thickness (into page) of the wedge is only one half times the the wedge itself[22]. Even for this small ratio two dimensional it provide very good results.

Another geometry that can be considered as two–dimensional is the cone (some referred to it as Taylor–Maccoll flow). Even though, the cone is a three–dimensional problem, the symmetrical nature of the cone creates a semi–2D problem. In this case there are no edge effects and the geometry dictates slightly different results. The mathematics is much more complicated but there are three solutions. As before, the first solution is thermodynamical unstable. Experimental and analytical work shows that the weak solution is the stable solution and a discussion is provided in the appendix of this chapter. As opposed to the weak shock, the strong shock is unstable, at least, for steady state and no known experiments showing that it exist can be found in the literature. All the literature, known to the author, reports that only a weak shock is possible.

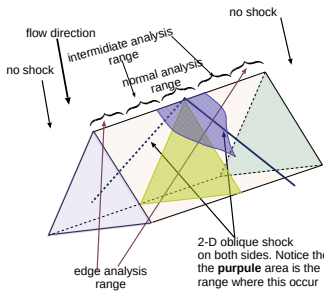


Fig. -14.12. Schematic of finite wedge with zero angle of attack. Notice that the purple area is only range where standard oblique shock analysis is applied.

14.4.7 Small δ "Weak Oblique shock"

This interest in this topic is mostly from an academic point of view. It is recommended that this issue be skipped and the time be devoted to other issues. The author is not aware of any single case in which this topic is used in real–world calculations. In fact, after the explicit analytical solution has been provided, studying this topic seems to come at the expense of other more important topics. However, the author admits that as long as there are instructors who examine their students on this issue, it should be covered in this book.

For small deflection angles, δ, and small normal upstream Mach numbers, $M_1 \sim 1+\epsilon$,

$$\tan\theta = \frac{1}{\sqrt{{M_1}^2 - 1}} \tag{14.63}$$

[21] At this stage, dimensional analysis is not completed. The author is not aware of any such analysis in literature. The common approach is to carry out numerical analysis. In spite of recent trends, for most engineering applications, a simple tool is sufficient for limit accuracy. Additionally, the numerical works require many times a "reality check."

[22] This information is according to Gary Settles which he provided the estimate only.

· · · under construction.

14.4.8 Close and Far Views of the Oblique Shock

In many cases, the close proximity view provides a continuous turning of the deflection angle, δ. Yet, the far view shows a sharp transition. The traditional approach to reconcile these two views is by suggesting that the far view shock is a collection of many small weak shocks (see Figure 14.13). At the local view close to the wall, the oblique shock is a weak "weak oblique" shock. From the far view, the oblique shock is an accumulation of many small (or again weak) "weak shocks." However, these small "shocks" are built or accumulate into a large and abrupt change (shock). In this theory, the boundary layer (B.L.) does not enter into the calculation. In reality, the boundary layer increases the zone where a continuous flow exists. The boundary layer reduces the upstream flow velocity and therefore the shock does not exist at close proximity to the wall. In larger distance from the wall, the shock becomes possible.

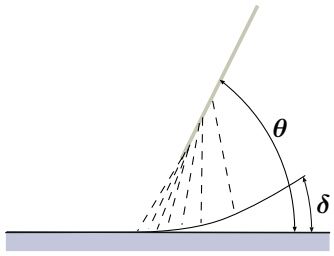


Fig. -14.13. A local and a far view of the oblique shock.

14.4.9 Maximum Value of Oblique shock

The maximum values are summarized in the following Table .

Table -14.1. Table of maximum values of the oblique Shock k=1.4

$\mathbf{M_x}$	$\mathbf{M_y}$	$\delta_{\mathbf{max}}$	$\theta_{\mathbf{max}}$
1.1000	0.97131	1.5152	76.2762
1.2000	0.95049	3.9442	71.9555
1.3000	0.93629	6.6621	69.3645
1.4000	0.92683	9.4272	67.7023
1.5000	0.92165	12.1127	66.5676
1.6000	0.91941	14.6515	65.7972
1.7000	0.91871	17.0119	65.3066
1.8000	0.91997	19.1833	64.9668
1.9000	0.92224	21.1675	64.7532

Table -14.1. Maximum values of oblique shock (continue) k=1.4

$\mathbf{M_x}$	$\mathbf{M_y}$	$\delta_{\mathbf{max}}$	$\theta_{\mathbf{max}}$
2.0000	0.92478	22.9735	64.6465
2.2000	0.93083	26.1028	64.6074
2.4000	0.93747	28.6814	64.6934
2.6000	0.94387	30.8137	64.8443
2.8000	0.94925	32.5875	65.0399
3.0000	0.95435	34.0734	65.2309
3.2000	0.95897	35.3275	65.4144
3.4000	0.96335	36.3934	65.5787
3.6000	0.96630	37.3059	65.7593
3.8000	0.96942	38.0922	65.9087
4.0000	0.97214	38.7739	66.0464
5.0000	0.98183	41.1177	66.5671
6.0000	0.98714	42.4398	66.9020
7.0000	0.99047	43.2546	67.1196
8.0000	0.99337	43.7908	67.2503
9.0000	0.99440	44.1619	67.3673
10.0000	0.99559	44.4290	67.4419

It must be noted that the calculations are for the perfect gas model. In some cases, this assumption might not be sufficient and different analysis is needed. Henderson and Menikoff[23] suggested a procedure to calculate the maximum deflection angle for arbitrary equation of state[24].

14.5 Detached Shock

When the mathematical quantity D becomes positive, for large deflection angle, there isn't a physical solution to an oblique shock. Since the flow "sees" the obstacle, the

[23] Henderson and Menikoff "Triple Shock Entropy Theorem" Journal of Fluid Mechanics 366 (1998) pp. 179–210.

[24] The effect of the equation of state on the maximum and other parameters at this state is unknown at this moment and there are more works underway.

only possible reaction is by a normal shock which occurs at some distance from the body. This shock is referred to as the detach shock. The detached shock's distance from the body is a complex analysis and should be left to graduate class and researchers in this area. Nevertheless, a graph and a general explanation to engineers is provided. Even though this topic has few applications, some might be used in certain situations which the author isn't aware of.

Analysis of the detached shock can be carried out by looking at a body with a round section moving in a supersonic flow (the absolute velocity isn't important for this discussion). Figure 14.14 exhibits a round–tip bullet with a detached shock. The distance of the detachment is determined to a large degree by the upstream Mach number. The zone A is zone where the flow must be subsonic because at the body the velocity must be zero (the no–slip condition). In such a case, the gas must go through a shock. While at zone C the flow must be supersonic. The weak oblique shock is predicted to flow around the cone. The flow in zone A has to go through some acceleration to became supersonic flow. The explanation to such a phenomenon is above the level of this book (where is the "throat" area question[25]. Yet, it can be explained as the subsonic is "sucked" into gas in zone **C**. Regardless of the explanation, these calculations can be summarized by the flowing equation

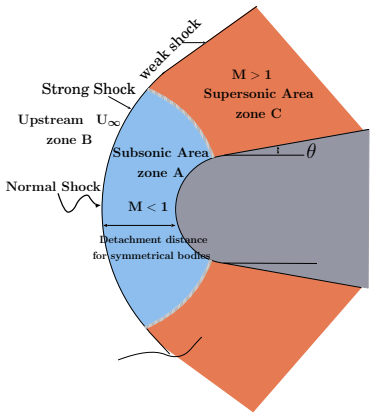


Fig. -14.14. The schematic for a round–tip bullet in a supersonic flow.

$$\frac{\text{detachment distance}}{\text{body thickness}} = \text{constant} \times (\theta - f(M_\infty)) \tag{14.64}$$

where $f(M_\infty)$ is a function of the upstream Mach number which tabulated in the literature.

The constant and the function are different for different geometries. As a general rule, the increase in the upstream Mach results in a decrease of the detachment distance. Larger shock results in a smaller detachment distance, or, alternatively, the flow becomes "blinder" to obstacles. Thus, this phenomenon has a larger impact for a relatively smaller supersonic flow.

14.5.1 Issues Related to the Maximum Deflection Angle

The issue of maximum deflection has a practical application aside from the obvious configuration used as a typical simple example. In the typical example, a wedge or a cone moves into a still medium or gas flows into it. If the deflection angle exceeds the maximum possible, a detached shock occurs. However, there are configurations in which

[25] See example 14.6.

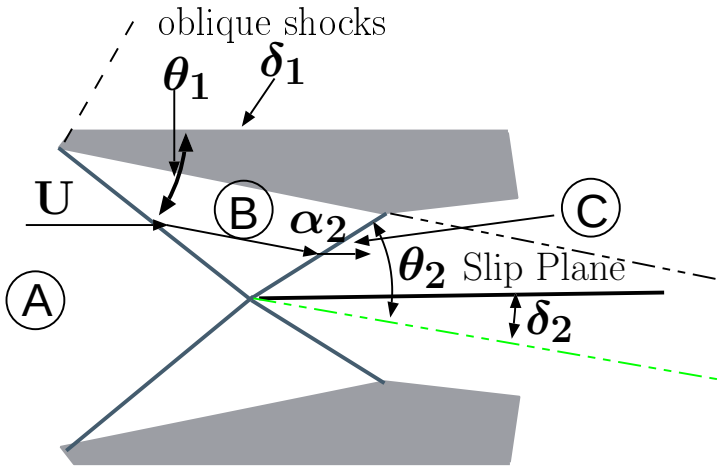


Fig. -14.15. The schematic for a symmetrical suction section with Mach reflection.

a detached shock occurs in design and engineers need to take it into consideration. Such configurations seem sometimes at first glance not related to the detached shock issue. Consider, for example, a symmetrical suction section in which the deflection angle is just between the maximum deflection angle and above half of the maximum deflection angle. In this situation, at least two oblique shocks occur and after their interaction is shown in Figure (14.15). No detached shock issues are raised when only the first oblique shock is considered. However, the second oblique shock complicates the situation and the second oblique shock can cause a detached shock. This situation is referred to in the scientific literature as the Mach reflection.

It can be observed that the maximum of the oblique shock for the perfect gas model depends only on the upstream Mach number i.e., for every upstream Mach number there is only one maximum deflection angle.

$$\delta_{max} = f(M_1) \tag{14.65}$$

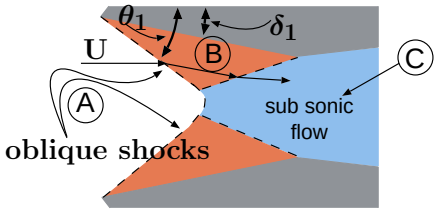


Fig. -14.16. The "detached" shock in a complicated configuration sometimes referred to as Mach reflection.

Additionally, it can be observed for a maximum oblique shock that a constant deflection angle decrease of the Mach number results in an increase of Mach angle (weak shock only) $M_1 > M_2 \Longrightarrow \theta_1 < \theta_2$. The Mach number decreases after every shock. Therefore, the maximum deflection angle decreases with a decrease the Mach number. Additionally, due to the symmetry a slip plane angle can be guessed to be parallel to original flow, hence $\delta_1 = \delta_2$. Thus, this situation causes the detached shock

to appear in the second oblique shock. This detached shock manifested itself in a form of curved shock (see Figure 14.16).

The analysis of this situation is logically very simple, yet the mathematics is somewhat complicated. The maximum deflection angle in this case is, as before, only a function of the upstream Mach number. The calculations for such a case can be carried out by several approaches. It seems that the most straightforward method is the following:

(a) Calculate M_{1B};

(b) Calculate the maximum deflection angle, θ_2, utilizing (14.36) equation

(c) Calculate the deflection angle, δ_2 utilizing equation (14.12)

(d) Use the deflection angle, $\delta_2 = \delta_1$ and the Mach number M_{1B} to calculate M_{1B}. Note that no maximum angle is achieved in this shock. Potto–GDC can be used to calculate this ratio.

This procedure can be extended to calculate the maximum incoming Mach number, M_1 by checking the relationship between the intermediate Mach number to M_1.

In discussing these issues, one must be aware that there are zones of dual solutions in which sharp shock line coexists with a curved line. In general, this zone increases as Mach number increases. For example, at Mach 5 this zone is 8.5°. For engineering purposes when the Mach number reaches this value, it can be ignored.

14.5.2 Oblique Shock Examples

Example 14.3:
Air flows at Mach number (M_1) or $M_x = 4$ is approaching a wedge. What is the maximum wedge angle at which the oblique shock can occur? If the wedge angle is 20°, calculate the weak, the strong Mach numbers, and the respective shock angles.

SOLUTION

The maximum wedge angle for ($M_x = 4$) D has to be equal to zero. The wedge angle that satisfies this requirement is by equation (14.28) (a side to the case proximity of $\delta = 0$). The maximum values are:

$\mathbf{M_x}$	$\mathbf{M_y}$	$\delta_{\mathbf{max}}$	$\theta_{\mathbf{max}}$
4.0000	0.97234	38.7738	66.0407

To obtain the results of the weak and the strong solutions either utilize the equation (14.28) or the GDC which yields the following results

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_{\mathbf{s}}$	$\theta_{\mathbf{w}}$	δ
4.0000	0.48523	2.5686	1.4635	0.56660	0.34907

End Solution

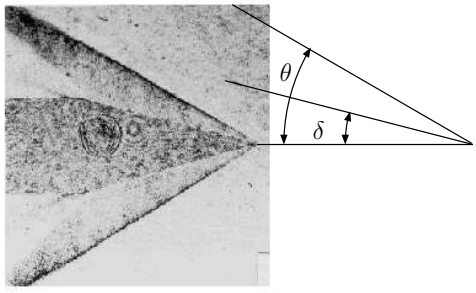


Fig. -14.17. Oblique shock occurs around a cone. This photo is courtesy of Dr. Grigory Toker, a Research Professor at Cuernavaco University of Mexico. According to his measurement, the cone half angle is 15° *and the Mach number is 2.2.*

Example 14.4:
A cone shown in Figure (14.17) *is exposed to supersonic flow and create an oblique shock. Is the shock shown in the photo weak or strong shock? Explain. Using the geometry provided in the photo, predict at which Mach number was the photo taken based on the assumption that the cone is a wedge.*

SOLUTION

The measurement shows that cone angle is 14.43° and the shock angle is 30.099°. With given two angles the solution can be obtained by utilizing equation (14.59) or the Potto-GDC.

$\mathbf{M_1}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.2318	0.56543	2.4522	71.0143	30.0990	14.4300	0.88737

Because the flow is around the cone it must be a weak shock. Even if the cone was a wedge, the shock would be weak because the maximum (transition to a strong shock) occurs at about 60°. Note that the Mach number is larger than the one predicted by the wedge.

End Solution

14.5.3 Application of Oblique Shock

One of the practical applications of the oblique shock is the design of an inlet suction for a supersonic flow. It is suggested that a series of weak shocks should replace one

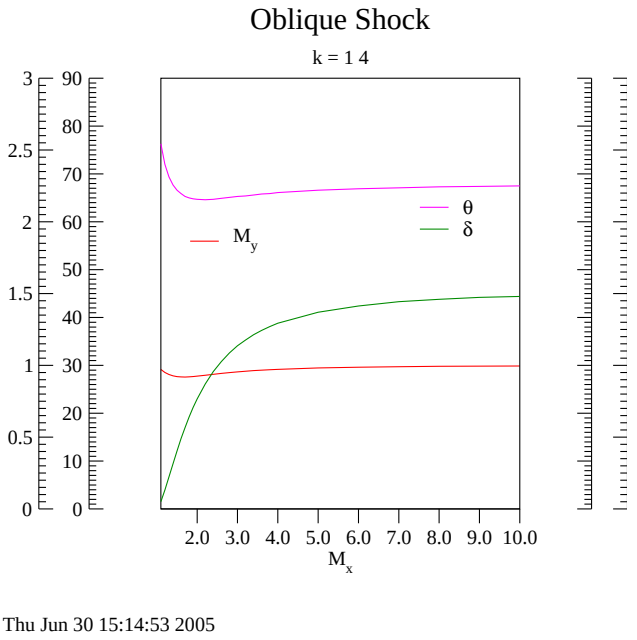


Fig. -14.18. Maximum values of the properties in an oblique shock.

normal shock to increase the efficiency (see Figure (14.19))[26]. Clearly, with a proper design, the flow can be brought to a subsonic flow just below $M = 1$. In such a case, there is less entropy production (less pressure loss). To illustrate the design significance of the oblique shock, the following example is provided.

Example 14.5:
The Section described in Figure 14.19 and 14.20 air is flowing into a suction section at $M = 2.0$, $P = 1.0[bar]$, and $T = 17°\mathrm{C}$. Compare the different conditions in the two different configurations. Assume that only a weak shock occurs.

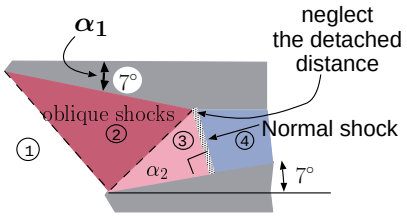


Fig. -14.20. Schematic for Example (14.5).

SOLUTION

[26] In fact, there is general proof that regardless to the equation of state (any kind of gas), the entropy is to be minimized through a series of oblique shocks rather than through a single normal shock. For details see Henderson and Menikoff "Triple Shock Entropy Theorem," Journal of Fluid Mechanics 366, (1998) pp. 179–210.

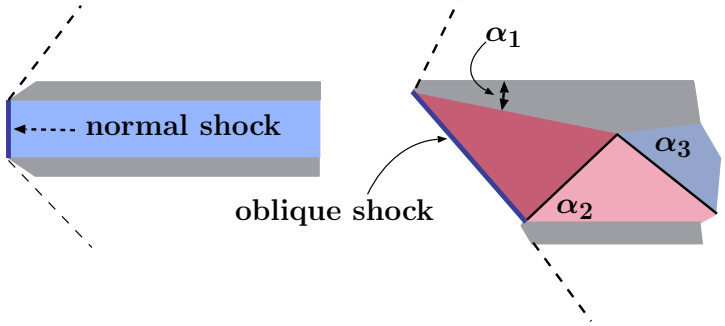


Fig. -14.19. Two variations of inlet suction for supersonic flow.

The first configuration is of a normal shock for which the results[27] are

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.0000	0.57735	1.6875	2.6667	4.5000	0.72087

In the oblique shock, the first angle shown is

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.0000	0.58974	1.7498	85.7021	36.2098	7.0000	0.99445

and the additional information by the minimal info in the Potto-GDC is

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.0000	1.7498	36.2098	7.0000	1.2485	1.1931	0.99445

In the new region, the new angle is $7^\circ + 7^\circ$ with new upstream Mach number of $M_x = 1.7498$ resulting in

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.7498	0.71761	1.2346	76.9831	51.5549	14.0000	0.96524

[27] The results in this example are obtained using the graphical interface of POTTO–GDC thus, no input explanation is given. In the past the input file was given but the graphical interface it is no longer needed.

And the additional information is

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.7498	1.5088	41.8770	7.0000	1.2626	1.1853	0.99549

An oblique shock is not possible and normal shock occurs. In such a case, the results are:

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.2346	0.82141	1.1497	1.4018	1.6116	0.98903

With two weak shock waves and a normal shock the total pressure loss is

$$\frac{P_{04}}{P_{01}} = \frac{P_{04}}{P_{03}}\frac{P_{03}}{P_{02}}\frac{P_{02}}{P_{01}} = 0.98903 \times 0.96524 \times 0.99445 = 0.9496$$

The static pressure ratio for the second case is

$$\frac{P_4}{P_1} = \frac{P_4}{P_3}\frac{P_3}{P_2}\frac{P_2}{P_1} = 1.6116 \times 1.2626 \times 1.285 = 2.6147$$

The loss in this case is much less than in a direct normal shock. In fact, the loss in the normal shock is above than 31% of the total pressure.

End Solution

Example 14.6:
A supersonic flow is approaching a very long two–dimensional bland wedge body and creates a detached shock at Mach 3.5 (see Figure 14.21). The half wedge angle is 10°. *What is the requited "throat" area ratio to achieve acceleration from the subsonic region to the supersonic region assuming the flow is one–dimensional?*

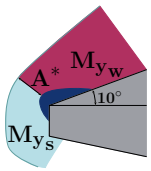


Fig. -14.21. Schematic for Example (14.6).

SOLUTION

The detached shock is a normal shock and the results are

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.5000	0.45115	3.3151	4.2609	14.1250	0.21295

Now utilizing the isentropic relationship for $k = 1.4$ yields

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^*}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
0.45115	0.96089	0.90506	1.4458	0.86966	1.2574

Thus the area ratio has to be 1.4458. Note that the pressure after the weak shock is irrelevant to the area ratio between the normal shock and the "throat" according to the standard nozzle analysis.

End Solution

Example 14.7:

The effects of a double wedge are explained in the government web site as shown in Figure 14.22. Adopt this description and assume that the turn of 6° is made of two equal angles of 3° (see Figure 14.22). Assume that there are no boundary layers and all the shocks are weak and straight. Perform the calculation for $M_1 = 3.0$. Find the required angle of shock BE. Then, explain why this description has internal conflict.

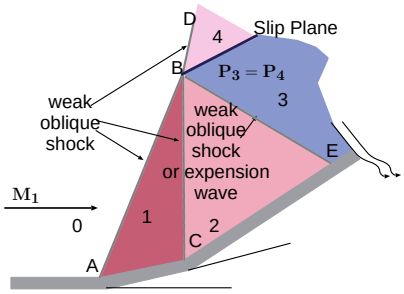


Fig. -14.22. Schematic of two angles turn with two weak shocks.

SOLUTION

The shock BD is an oblique shock with a response to a total turn of 6°. The conditions for this shock are:

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.0000	0.48013	2.7008	87.8807	23.9356	6.0000	0.99105

The transition for shock AB is

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.0000	0.47641	2.8482	88.9476	21.5990	3.0000	0.99879

For the shock BC the results are

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.8482	0.48610	2.7049	88.8912	22.7080	3.0000	0.99894

And the isentropic relationships for $M = 2.7049,\ 2.7008$ are

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$
2.7049	0.40596	0.10500	3.1978	0.04263	0.13632
2.7008	0.40669	0.10548	3.1854	0.04290	0.13665

The combined shocks AB and BC provide the base of calculating the total pressure ratio at zone 3. The total pressure ratio at zone 2 is

$$\frac{P_{02}}{P_{00}} = \frac{P_{02}}{P_{01}}\frac{P_{01}}{P_{00}} = 0.99894 \times 0.99879 = 0.997731283$$

On the other hand, the pressure at 4 has to be

$$\frac{P_4}{P_{01}} = \frac{P_4}{P_{04}}\frac{P_{04}}{P_{01}} = 0.04290 \times 0.99105 = 0.042516045$$

The static pressure at zone 4 and zone 3 have to match according to the government suggestion hence, the angle for BE shock which cause this pressure ratio needs to be found. To do that, check whether the pressure at 2 is above or below or above the pressure (ratio) in zone 4.

$$\frac{P_2}{P_{02}} = \frac{P_{02}}{P_{00}}\frac{P_2}{P_{02}} = 0.997731283 \times 0.04263 = 0.042436789$$

Since $\frac{P_2}{P_{02}} < \frac{P_4}{P_{01}}$ a weak shock must occur to increase the static pressure (see Figure 6.4). The increase has to be

$$P_3/P_2 = 0.042516045/0.042436789 = 1.001867743$$

To achieve this kind of pressure ratio the perpendicular component has to be

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
1.0008	0.99920	1.0005	1.0013	1.0019	1.00000

The shock angle, θ can be calculated from

$$\theta = \sin^{-1} 1.0008/2.7049 = 21.715320879^\circ$$

The deflection angle for such shock angle with Mach number is

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.7049	0.49525	2.7037	0.0	21.72	0.026233	1.00000

From the last calculation it is clear that the government proposed schematic of the double wedge is in conflict with the boundary condition. The flow in zone 3 will flow into the wall in about 2.7°. In reality the flow of double wedge will produce a curved shock surface with several zones. Only when the flow is far away from the double wedge, the flow behaves as only one theoretical angle of 6° exist.

End Solution

Example 14.8:
Calculate the flow deflection angle and other parameters downstream when the Mach angle is 34° *and* $P_1 = 3[bar]$, $T_1 = 27^\circ\mathrm{C}$, *and* $U_1 = 1000m/sec$. *Assume* $k = 1.4$ *and* $R = 287J/KgK$.

SOLUTION

The Mach angle of 34° is below maximum deflection which means that it is a weak shock. Yet, the Upstream Mach number, M_1, has to be determined

$$M_1 = \frac{U_1}{\sqrt{k\,R\,T}} = \frac{1000}{1.4 \times 287 \times 300} = 2.88$$

Using this Mach number and the Mach deflection in either using the Table or the figure or POTTO-GDC results in

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.8800	0.48269	2.1280	0.0	34.00	15.78	0.89127

The relationship for the temperature and pressure can be obtained by using equation (14.15) and (14.13) or simply converting the M_1 to perpendicular component.

$$M_{1n} = M_1 * \sin\theta = 2.88\sin(34.0) = 1.61$$

From the Table (6.2) or GDC the following can be obtained.

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
1.6100	0.66545	1.3949	2.0485	2.8575	0.89145

The temperature ratio combined upstream temperature yield

$$T_2 = 1.3949 \times 300 \sim 418.5K$$

and the same for the pressure

$$P_2 = 2.8575 \times 3 = 8.57[bar]$$

And the velocity

$$U_{n2} = M_{y_w}\sqrt{k\,R\,T} = 2.128\sqrt{1.4 \times 287 \times 418.5} = 872.6[m/sec]$$

End Solution

Example 14.9:
For Mach number 2.5 and wedge with a total angle of 22°, calculate the ratio of the stagnation pressure.

SOLUTION

Utilizing GDC for Mach number 2.5 and the angle of 11° results in

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.5000	0.53431	2.0443	85.0995	32.8124	11.0000	0.96873

End Solution

Example 14.10:
What is the maximum pressure ratio that can be obtained on wedge when the gas is flowing in 2.5 Mach without any close boundaries? Would it make any difference if the wedge was flowing into the air? If so, what is the difference?

SOLUTION

It has to be recognized that without any other boundary condition, the shock is weak shock. For a weak shock the maximum pressure ratio is obtained at the deflection point because it is closest to a normal shock. To obtain the maximum point for 2.5 Mach number, either use the Maximum Deflection Mach number's equation or the Potto–GDC

$\mathbf{M_x}$	$\mathbf{M_{y_{max}}}$	$\theta_\mathbf{max}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
2.5000	0.94021	64.7822	29.7974	4.3573	2.6854	0.60027

In these calculations, Maximum Deflection Mach's equation was used to calculate the normal component of the upstream, then the Mach angle was calculated using the geometrical relationship of $\theta = \sin^{-1} M_{1n}/M_1$. With these two quantities, utilizing equation (14.12) the deflection angle, δ, is obtained.

End Solution

Example 14.11:

Consider the schematic shown in the following figure. Assume that the upstream Mach number is 4 and the deflection angle is $\delta = 15^\circ$. Compute the pressure ratio and the temperature ratio after the second shock (sometimes referred to as the reflective shock while the first shock is called the incidental shock).

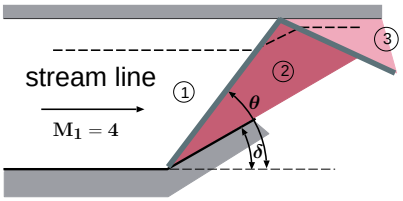


Fig. -14.23. Schematic for Example (14.11).

SOLUTION

This kind of problem is essentially two wedges placed in a certain geometry. It is clear that the flow must be parallel to the wall. For the first shock, the upstream Mach number is known together with deflection angle. Utilizing the table or the Potto–GDC, the following can be obtained:

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
4.0000	0.46152	2.9290	85.5851	27.0629	15.0000	0.80382

And the additional information by using minimal information ratio button in Potto–GDC is

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
4.0000	2.9290	27.0629	15.0000	1.7985	1.7344	0.80382

With a Mach number of $M = 2.929$, the second deflection angle is also 15°. With these values the following can be obtained:

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.9290	0.51367	2.2028	84.2808	32.7822	15.0000	0.90041

and the additional information is

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.9290	2.2028	32.7822	15.0000	1.6695	1.5764	0.90041

With the combined tables the ratios can be easily calculated. Note that hand calculations requires endless time looking up graphical representation of the solution. Utilizing the POTTO–GDC which provides a solution in just a few clicks.

$$\frac{P_1}{P_3} = \frac{P_1}{P_2}\frac{P_2}{P_3} = 1.7985 \times 1.6695 = 3.0026$$

$$\frac{T_1}{T_3} = \frac{T_1}{T_2}\frac{T_2}{T_3} = 1.7344 \times 1.5764 = 2.632$$

End Solution

Example 14.12:
A similar example as before but here Mach angle is 29° and Mach number is 2.85. Again calculate the downstream ratios after the second shock and the deflection angle.

SOLUTION

Here the Mach number and the Mach angle are given. With these pieces of information by utilizing the Potto-GDC the following is obtained:

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.8500	0.48469	2.3575	0.0	29.00	10.51	0.96263

and the additional information by utilizing the minimal info button in GDC provides

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.8500	2.3575	29.0000	10.5131	1.4089	1.3582	0.96263

With the deflection angle of $\delta = 10.51$ the so called reflective shock gives the following information

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.3575	0.54894	1.9419	84.9398	34.0590	10.5100	0.97569

and the additional information of

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
2.3575	1.9419	34.0590	10.5100	1.3984	1.3268	0.97569

$$\frac{P_1}{P_3} = \frac{P_1}{P_2}\frac{P_2}{P_3} = 1.4089 \times 1.3984 \sim 1.97$$
$$\frac{T_1}{T_3} = \frac{T_1}{T_2}\frac{T_2}{T_3} = 1.3582 \times 1.3268 \sim 1.8021$$

End Solution

Example 14.13:
Compare a direct normal shock to oblique shock with a normal shock. Where will the total pressure loss (entropy) be larger? Assume that upstream Mach number is 5 and the first oblique shock has Mach angle of 30°*. What is the deflection angle in this case?*

SOLUTION

For the normal shock the results are

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
5.0000	0.41523	5.8000	5.0000	29.0000	0.06172

While the results for the oblique shock are

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
5.0000	0.41523	3.0058	0.0	30.00	20.17	0.49901

And the additional information is

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
5.0000	3.0058	30.0000	20.1736	2.6375	2.5141	0.49901

The normal shock that follows this oblique is

$\mathbf{M_x}$	$\mathbf{M_y}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\rho_\mathbf{y}}{\rho_\mathbf{x}}$	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
3.0058	0.47485	2.6858	3.8625	10.3740	0.32671

The pressure ratios of the oblique shock with normal shock is the total shock in the second case.

$$\frac{P_1}{P_3} = \frac{P_1}{P_2}\frac{P_2}{P_3} = 2.6375 \times 10.374 \sim 27.36$$
$$\frac{T_1}{T_3} = \frac{T_1}{T_2}\frac{T_2}{T_3} = 2.5141 \times 2.6858 \sim 6.75$$

Note the static pressure raised is less than the combination shocks as compared to the normal shock but the total pressure has the opposite result.

End Solution

Example 14.14:

A flow in a tunnel ends up with two deflection angles from both sides (see the following Figure 14.14). For upstream Mach number of 5 and deflection angle of 12° *and* 15°*, calculate the pressure at zones 3 and 4 based on the assumption that the slip plane is half of the difference between the two deflection angles. Based on these calculations, explain whether the slip angle is larger or smaller than the difference of the deflection angle.*

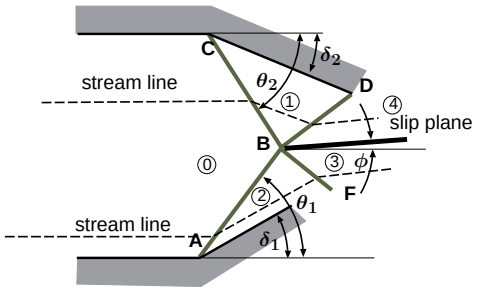


Fig. -14.24. Illustration for Example (14.14).

SOLUTION

The first two zones immediately after are computed using the same techniques that were developed and discussed earlier.

For the first direction of 15° and Mach number $=$5.

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
5.0000	0.43914	3.5040	86.0739	24.3217	15.0000	0.69317

And the additional conditions are

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
5.0000	3.5040	24.3217	15.0000	1.9791	1.9238	0.69317

For the second direction of 12° and Mach number $=$5.

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0y}}}{\mathbf{P_{0x}}}$
5.0000	0.43016	3.8006	86.9122	21.2845	12.0000	0.80600

And the additional conditions are

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
5.0000	3.8006	21.2845	12.0000	1.6963	1.6625	0.80600

The conditions in zone 4 and zone 3 have two things that are equal. They are the pressure and the velocity direction. It has to be noticed that the velocity magnitudes in zone 3 and 4 do not have to be equal. This non–continuous velocity profile can occur in our model because it is assumed that fluid is non–viscous.

If the two sides were equal because of symmetry the slip angle is also zero. It is to say, for the analysis, that only one deflection angle exist. For the two different deflection angles, the slip angle has two extreme cases. The first case is where match lower deflection angle and second is to match the higher deflection angle. In this case, it is assumed that the slip angle moves half of the angle to satisfy both of the deflection angles (first approximation). Under this assumption the conditions in zone 3 are solved by looking at the deflection angle of $12^\circ + 1.5^\circ = 13.5^\circ$ which results in

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.5040	0.47413	2.6986	85.6819	27.6668	13.5000	0.88496

with the additional information

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.5040	2.6986	27.6668	13.5000	1.6247	1.5656	0.88496

And in zone 4 the conditions are due to deflection angle of 13.5° and Mach 3.8006

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.8006	0.46259	2.9035	85.9316	26.3226	13.5000	0.86179

with the additional information

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.8006	2.9035	26.3226	13.5000	1.6577	1.6038	0.86179

From these tables the pressure ratio at zone 3 and 4 can be calculated

$$\frac{P_3}{P_4} = \frac{P_3}{P_2}\frac{P_2}{P_0}\frac{P_0}{P_1}\frac{P_1}{P_4} = 1.6247 \times 1.9791 \frac{1}{1.6963}\frac{1}{1.6038} \sim 1.18192$$

To reduce the pressure ratio the deflection angle has to be reduced (remember that at weak weak shock almost no pressure change). Thus, the pressure at zone 3 has to be reduced. To reduce the pressure the angle of slip plane has to increase from $1.5\circ$ to a larger number.

End Solution

Example 14.15:
The previous example gave rise to another question on the order of the deflection angles. Consider the same values as previous analysis, will the oblique shock with first angle of 15° *and then* 12° *or opposite order make a difference* ($M = 5$)*? If not what order will make a bigger entropy production or pressure loss? (No general proof is needed).*

SOLUTION

Waiting for the solution

End Solution

14.5.4 Optimization of Suction Section Design

Under heavy construction please ignore

The question raised is what is the optimum design for inlet suction unit? There are several considerations that have to be taken into account besides supersonic flow which includes for example the material strength consideration and the operation factors.

The optimum deflection angle is a function of the Mach number range in which the suction section is operated in. There are researchers which suggest that the numerical work is the solution.

14.5.5 Retouch of Shock Drag or Wave Drag

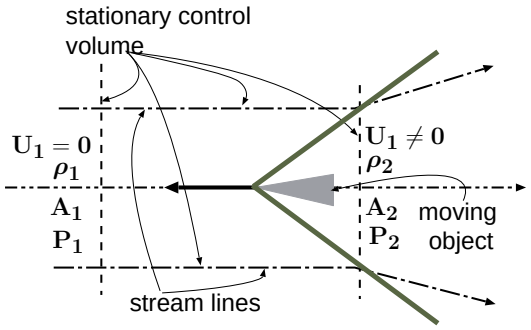


Fig. -14.25. The diagram that explains the shock drag effects of a moving shock considering the oblique shock effects.

Since it was established that the common explanation is erroneous and the steam lines are bending/changing direction when they touching the oblique shock (compare with figure (6.7)). The correct explanation is that increase of the momentum into control volume is either requires increase of the force and/or results in acceleration of gas. So, what is the effects of the oblique shock on the Shock Drag? Figure (14.25) exhibits schematic of the oblique shock which show clearly that stream lines are bended. There two main points that should be discussed in this context are the additional effects and infinite/final structure. The additional effects are the mass start to have a vertical component. The vertical component one hand increase the energy needed and thus increase need to move the body (larger shock drag) (note the there is a zero momentum net change for symmetrical bodies.). However, the oblique shock reduces the normal component that undergoes the shock and hence the total shock drag is reduced. The oblique shock creates a finite amount of drag (momentum and energy lost) while a normal shock as indirectly implied in the common explanation creates de facto situation where the shock grows to be infinite which of course impossible. It should be noted that, oblique shock becomes less "oblique" and more parallel when other effects start to kick in.

14.6 Summary

As with normal shock, the oblique shock with upstream Mach number, M_1 is always greater than 1. However, in oblique, as oppose to the normal shock, the downstream Mach number, M_2 could be larger or smaller then 1. The perpendicular component of the downstream Mach number, M_{1_n} is always smaller than one (1). Given M_1 and the deflection angle, δ there could be three solutions: the first one is the "impossible" solution in the case where D is negative, second is weak shock, and third is a strong shock. When D is positive there is no physical solution and only normal shock exist. When D is equal to zero, a special case is created because the weak and strong solutions are equal (for large deflection angle). When $D > 0$, for large deflection angle, there is a possibility of no two–dimensional solution resulting in a detached shock case.

14.7 Appendix: Oblique Shock Stability Analysis

The stability analysis is an analysis which answers the question of what happens if for some reason, the situation moves away from the expected solution. If the answer turns out to be that the situation will return to its original state then it is referred to as the stable situation. On the other hand, if the answer is negative, then the situation is referred to as unstable. An example to this situation, is a ball shown in the Figure (14.27). Instinctively, the stable and unstable can be recognized. There is also the situation where the ball is between

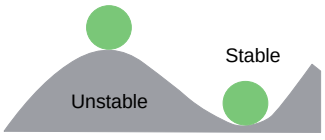


Fig. -14.27. Typical examples of unstable and stable situations.

the stable and unstable situations when the ball is on a plane field which is referred to as the neutrally stable. In the same manner, the analysis for the oblique shock wave is carried out. The only difference is that here, there are more than one parameter that can be changed, for example, the shock angle, deflection angle, and upstream Mach number. In this example only the weak solution is explained. The similar analysis can be applied to strong shock. Yet, in that analysis it has to be remembered that when the flow becomes subsonic the equation changes from hyperbolic to an elliptic equation. This change complicates the explanation and is omitted in this section. Of course, in the analysis the strong shock results in an elliptic solution (or region) as opposed to a hyperbolic in weak shock. As results, the discussion is more complicated but similar analysis can be applied to the strong shock.

The change in the inclination angle results in a different upstream Mach number and a different pressure. On the other hand, to maintain the same direction stream lines, the virtual change in the deflection angle has to be in the opposite direction of the change of the shock angle. The change is determined from the solution provided before or from the approximation (14.62).

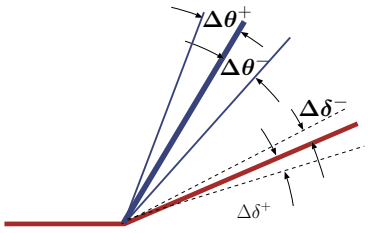
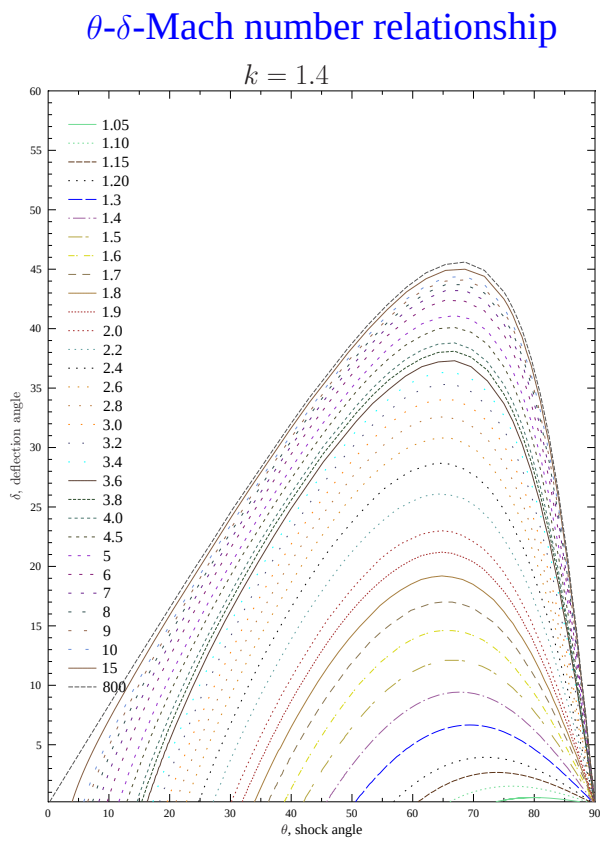


Fig. -14.28. The schematic of stability analysis for oblique shock.

$$\Delta\theta = \frac{k+1}{2}\Delta\delta \qquad (14.66)$$

Equation (14.66) can be applied for either positive, $\Delta\theta^+$ or negative $\Delta\theta^-$ values. The pressure difference at the wall becomes a negative increment which tends to pull the shock angle to the opposite direction. The opposite happens when the deflection increment becomes negative, the deflection angle becomes positive which increases the pressure at the wall. Thus, the weak shock is stable.

Please note that this analysis doesn't apply to the case of the close proximity of the $\delta = 0$. In fact, the shock wave is unstable according to this analysis to one direction but stable to the other direction. Yet, it must be pointed out that it doesn't mean that the flow is unstable but rather that the model is incorrect. There isn't any known experimental evidence to show that flow is unstable for $\delta = 0$.



December 4, 2007

Fig. -14.26. The relationship between the shock wave angle, θ and deflection angle, δ, and Mach number for k=1.4. This figure was generate with GDC under command ./obliqueFigure 1.4. Variety of these figures can be found in the biggest gas tables in the world provided separately in Potto Project.

CHAPTER 15

Prandtl-Meyer Function

15.1 Introduction

As discussed in Chapter 14 when the deflection turns to the opposite direction of the flow, the flow accelerates to match the boundary condition. The transition, as opposed to the oblique shock, is smooth, without any jump in properties. Here because of the tradition, the deflection angle is denoted as a positive when it is away from the flow (see Figure 15.1). In a somewhat a similar concept to oblique shock there exists a "detachment" point above which this model breaks and another model has to be implemented. Yet, when this model breaks down, the flow becomes complicated, flow separation occurs, and no known simple model can describe the situation. As opposed to the oblique shock, there is no limitation for the Prandtl-Meyer function to approach zero. Yet, for very small angles, because of imperfections of the wall and the boundary layer, it has to be assumed to be insignificant.

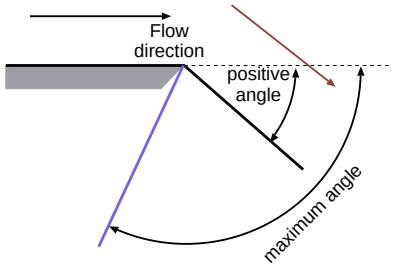


Fig. -15.1. The definition of the angle for the Prandtl–Meyer function.

Supersonic expansion and isentropic compression (Prandtl-Meyer function), are an extension of the Mach line concept. The Mach line shows that a disturbance in a field of supersonic flow moves in an angle of μ, which is defined as (as shown in Figure 15.2)

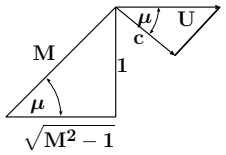


Fig. -15.2. The angles of the Mach line triangle.

$$\mu = \sin^{-1}\left(\frac{1}{M}\right) \tag{15.1}$$

or

$$\mu = \tan^{-1} \frac{1}{\sqrt{M^1 - 1}} \tag{15.2}$$

A Mach line results because of a small disturbance in the wall contour. This Mach line is assumed to be a result of the positive angle. The reason that a "negative" angle is not applicable is that the coalescing of the small Mach wave which results in a shock wave. However, no shock is created from many small positive angles.

The Mach line is the chief line in the analysis because of the wall contour shape information propagates along this line. Once the contour is changed, the flow direction will change to fit the wall. This direction change results in a change of the flow properties, and it is assumed here to be isotropic for a positive angle. This assumption, as it turns out, is close to reality. In this chapter, a discussion on the relationship between the flow properties and the flow direction is presented.

15.2 Geometrical Explanation

The change in the flow direction is assume to be result of the change in the tangential component. Hence, the total Mach number increases. Therefore, the Mach angle increase and result in a change in the direction of the flow. The velocity component in the direction of the Mach line is assumed to be constant to satisfy the assumption that the change is a result of the contour only. Later, this assumption will be examined. The typical simplifications for geometrical functions are used:

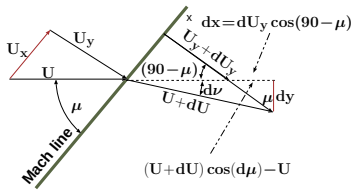


Fig. -15.3. The schematic of the turning flow.

$$\begin{aligned} d\nu &\sim \sin(d\nu); \\ \cos(d\nu) &\sim 1 \end{aligned} \tag{15.3}$$

These simplifications are the core reasons why the change occurs only in the perpendicular direction ($d\nu << 1$). The change of the velocity in the flow direction, dx is

$$dx = (U + dU)\cos\nu - U = dU \tag{15.4}$$

In the same manner, the velocity perpendicular to the flow, dy, is

$$dy = (U + dU)\sin(d\nu) = U d\nu \tag{15.5}$$

The $\tan\mu$ is the ratio of dy/dx (see Figure (15.3))

$$\tan\mu = \frac{dx}{dy} = \frac{dU}{U d\nu} \tag{15.6}$$

The ratio dU/U was shown to be

$$\frac{dU}{U} = \frac{dM^2}{2M^2\left(1+\frac{k-1}{2}M^2\right)} \tag{15.7}$$

Combining equations (15.6) and (15.7) transforms it into

$$d\nu = -\frac{\sqrt{M^2-1}dM^2}{2M^2\left(1+\frac{k-1}{2}M^2\right)} \tag{15.8}$$

After integration of equation (15.8) becomes

Turnning Angle

$$\nu(M) = -\sqrt{\frac{k+1}{k-1}}\ \tan^{-1}\sqrt{\frac{k-1}{k+1}\left(M^2-1\right)} + \tan^{-1}\sqrt{\left(M^2-1\right)} + constant \tag{15.9}$$

The constant can be chosen in a such a way that $\nu = 0$ at $M = 1$.

15.2.1 Alternative Approach to Governing Equations

In the previous section, a simplified version was derived based on geometrical arguments. In this section, a more rigorous explanation is provided. It must be recognized that here the cylindrical coordinates are advantageous because the flow turns around a single point.

For this coordinate system, the mass conservation can be written as

$$\frac{\partial\left(\rho r U_r\right)}{\partial r} + \frac{\partial\left(\rho U_\theta\right)}{\partial\theta} = 0 \tag{15.10}$$

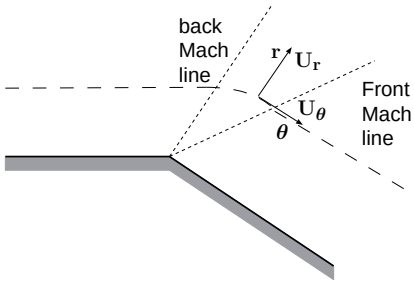


Fig. -15.4. The schematic of the coordinate based on the mathematical description.

The momentum equations are expressed as

$$U_r\frac{\partial U_r}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_r}{\partial\theta} - \frac{{U_\theta}^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{c^2}{\rho}\frac{\partial\rho}{\partial r} \tag{15.11}$$

and

$$U_r\frac{\partial U_\theta}{\partial r} + \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial\theta} - \frac{U_\theta U_r}{r} = -\frac{1}{r\rho}\frac{\partial P}{\partial\theta} = -\frac{c^2}{r\rho}\frac{\partial\rho}{\partial\theta} \tag{15.12}$$

If the assumption is that the flow isn't a function of the radius, r, then all the derivatives with respect to the radius will vanish. One has to remember that when r enters to the

function, like the first term in the mass equation, the derivative isn't zero. Hence, the mass equation is reduced to

$$\rho U_r + \frac{\partial \left(\rho U_\theta\right)}{\partial \theta} = 0 \tag{15.13}$$

Equation (15.13) can be rearranged as transformed into

$$-\frac{1}{U_\theta}\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) = \frac{1}{\rho}\frac{\partial \rho}{\partial \theta} \tag{15.14}$$

The momentum equations now obtain the form of

$$\begin{aligned} \frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta} - \frac{{U_\theta}^2}{r} &= 0 \\ U_\theta\left(\frac{\partial U_r}{\partial \theta} - U_\theta\right) &= 0 \end{aligned} \tag{15.15}$$

$$\begin{aligned} \frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta} - \frac{U_\theta U_r}{r} &= -\frac{c^2}{r\rho}\frac{\partial \rho}{\partial \theta} \\ U_\theta\left(\frac{\partial U_\theta}{\partial \theta} - U_r\right) &= -\frac{c^2}{\rho}\frac{\partial \rho}{\partial \theta} \end{aligned} \tag{15.16}$$

Substituting the term $\frac{1}{\rho}\frac{\partial \rho}{\partial \theta}$ from equation (15.14) into equation (15.16) results in

$$U_\theta\left(\frac{\partial U_\theta}{\partial \theta} - U_r\right) = \frac{c^2}{U_\theta}\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) \tag{15.17}$$

or

$${U_\theta}^2\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) = c^2\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) \tag{15.18}$$

And an additional rearrangement results in

$$\left(c^2 - {U_\theta}^2\right)\left(U_r + \frac{\partial U_\theta}{\partial \theta}\right) = 0 \tag{15.19}$$

From equation (15.19) it follows that

$$U_\theta = c \tag{15.20}$$

It is remarkable that the tangential velocity at every turn is at the speed of sound! It must be pointed out that the total velocity isn't at the speed of sound, but only the tangential component. In fact, based on the definition of the Mach angle, the component shown in Figure (15.3) under U_y is equal to the speed of sound, $M = 1$.

After some additional rearrangement, equation (15.15) becomes

$$\frac{U_\theta}{r}\left(\frac{\partial U_r}{\partial \theta} - U_\theta\right) = 0 \tag{15.21}$$

If r isn't approaching infinity, ∞ and since $U_\theta \neq 0$ leads to

$$\frac{\partial U_r}{\partial \theta} = U_\theta \tag{15.22}$$

In the literature, these results are associated with the characteristic line. This analysis can be also applied to the same equation when they are normalized by Mach number. However, the non–dimensionalization can be applied at this stage as well.

The energy equation for any point on a stream line is

$$h(\theta) + \frac{{U_\theta}^2 + {U_r}^2}{2} = h_0 \tag{15.23}$$

Enthalpy in perfect gas with a constant specific heat, k, is

$$h(\theta) = C_p\, T = C_p\, \frac{R}{R} T = \frac{1}{(k-1)}\, \overbrace{\frac{C_p}{C_v}}^{k}\, \overbrace{R\, T}^{c(\theta)^2} = \frac{c^2}{k-1} \tag{15.24}$$

and substituting this equality, equation (15.24), into equation (15.23) results in

$$\frac{c^2}{k-1} + \frac{{U_\theta}^2 + {U_r}^2}{2} = h_0 \tag{15.25}$$

Utilizing equation (15.20) for the speed of sound and substituting equation (15.22) which is the radial velocity transforms equation (15.25) into

$$\frac{\left(\dfrac{\partial U_r}{\partial \theta}\right)^2}{k-1} + \frac{\left(\dfrac{\partial U_r}{\partial \theta}\right)^2 + {U_r}^2}{2} = h_0 \tag{15.26}$$

After some rearrangement, equation (15.26) becomes

$$\frac{k+1}{k-1}\left(\frac{\partial U_r}{\partial \theta}\right)^2 + {U_r}^2 = 2h_0 \tag{15.27}$$

Note that U_r must be positive. The solution of the differential equation (15.27) incorporating the constant becomes

$$U_r = \sqrt{2h_0}\sin\left(\theta\sqrt{\frac{k-1}{k+1}}\right) \tag{15.28}$$

which satisfies equation (15.27) because $\sin^2\theta + \cos^2\theta = 1$. The arbitrary constant in equation (15.28) is chosen such that $U_r(\theta = 0) = 0$. The tangential velocity obtains the form

$$U_\theta = c = \frac{\partial U_r}{\partial \theta} = \sqrt{\frac{k-1}{k+1}}\sqrt{2\,h_0}\;\cos\left(\theta\sqrt{\frac{k-1}{k+1}}\right) \tag{15.29}$$

The Mach number in the turning area is

$$M^2 = \frac{{U_\theta}^2 + {U_r}^2}{c^2} = \frac{{U_\theta}^2 + {U_r}^2}{{U_\theta}^2} = 1 + \left(\frac{U_r}{U_\theta}\right)^2 \tag{15.30}$$

Now utilizing the expression that was obtained for U_r and U_θ equations (15.29) and (15.28) results for the Mach number is

$$M^2 = 1 + \frac{k+1}{k-1}\tan^2\left(\theta\sqrt{\frac{k-1}{k+1}}\right) \tag{15.31}$$

or the reverse function for θ is

Reverse Angle

$$\theta = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}}\left(M^2-1\right)\right) \tag{15.32}$$

What happens when the upstream Mach number is not 1? That is when the initial condition for the turning angle doesn't start with $M = 1$ but is already at a different angle. The upstream Mach number is denoted in this segment as $M_{starting}$. For this upstream Mach number (see Figure (15.2))

$$\tan\nu = \sqrt{{M_{starting}}^2 - 1} \tag{15.33}$$

The deflection angle ν, has to match to the definition of the angle that is chosen here ($\theta = 0$ when $M = 1$), so

$$\nu(M) = \theta(M) - \theta(M_{starting}) \tag{15.34}$$

Deflection Angle

$$\nu(M) = \sqrt{\frac{k+1}{k-1}}\tan^{-1}\left(\sqrt{\frac{k-1}{k+1}}\sqrt{M^2-1}\right) - \tan^{-1}\sqrt{M^2-1} \tag{15.35}$$

These relationships are plotted in Figure (15.6).

15.2.2 Comparison And Limitations between the Two Approaches

The two models produce exactly the same results, but the assumptions for the construction of these models are different. In the geometrical model, the assumption is that the velocity change in the radial direction is zero. In the rigorous model, it was assumed that radial velocity is only a function of θ. The statement for the construction of the geometrical model can be improved by assuming that the frame of reference is moving radially in a constant velocity.

Regardless of the assumptions that were used in the construction of these models, the fact remains that there is a radial velocity at $U_r(r=0) = constant$. At this point $(r=0)$ these models fail to satisfy the boundary conditions and something else happens there. On top of the complication of the turning point, the question of boundary layer arises. For example, how did the gas accelerate to above the speed of sound when there is no nozzle (where is the nozzle?)? These questions are of interest in engineering but are beyond the scope of this book (at least at this stage). Normally, the author recommends that this function be used everywhere beyond 2-4 the thickness of the boundary layer based on the upstream length.

In fact, analysis of design commonly used in the industry and even questions posted to students show that many assume that the turning point can be sharp. At a small Mach number, $(1+\epsilon)$ the radial velocity is small ϵ. However, an increase in the Mach number can result in a very significant radial velocity. The radial velocity is "fed" through the reduction of the density. Aside from its close proximity to turning point, mass balance is maintained by the reduction of the density. Thus, some researchers recommend that, in many instances, the sharp point should be replaced by a smoother transition.

15.3 The Maximum Turning Angle

The maximum turning angle is obtained when the starting Mach number is 1 and the end Mach number approaches infinity. In this case, Prandtl–Meyer function becomes

Maximum Turning Angle

$$\nu_\infty = \frac{\pi}{2}\left[\sqrt{\frac{k+1}{k-1}} - 1\right] \tag{15.36}$$

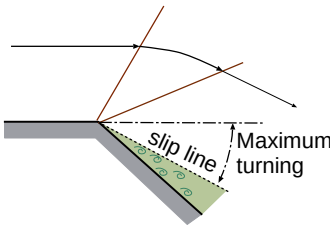


Fig. -15.5. Expansion of Prandtl-Meyer function when it exceeds the maximum angle.

The maximum of the deflection point and the maximum turning point are only a function of the specific heat ratios. However, the maximum turning angle is much larger than the maximum deflection point because the process is isentropic.

What happens when the deflection angel exceeds the maximum angle? The flow in this case behaves as if there is almost a maximum angle and in that region beyond the flow will became vortex street see Figure (15.5)

15.4 The Working Equations for the Prandtl-Meyer Function

The change in the deflection angle is calculated by

$$\nu_2 - \nu_1 = \nu(M_2) - \nu(M_1) \tag{15.37}$$

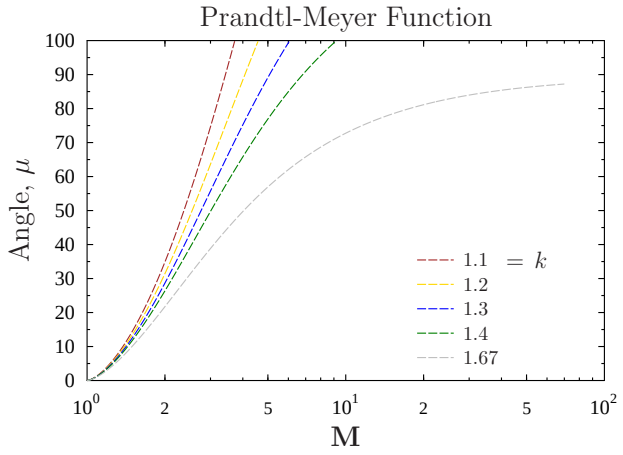


Fig. -15.6. The angle as a function of the Mach number and spesfic heat.

15.5 d'Alembert's Paradox

In ideal inviscid incompressible flows, the movement of body does not encounter any resistance. This result is known as d'Alembert's Paradox, and this paradox is examined here.

Supposed that a two–dimensional diamond–shape body is stationed in a supersonic flow as shown in Figure (15.7). Again, it is assumed that the fluid is inviscid. The net force in flow direction, the drag, is

$$D = 2\left(\frac{w}{2}\left(P_2 - P_4\right)\right) = w\left(P_2 - P_4\right) \tag{15.38}$$

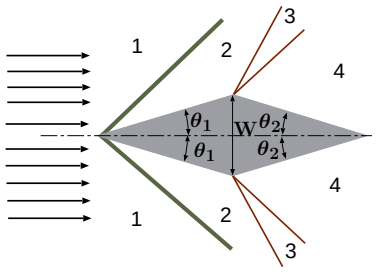


Fig. -15.7. A simplified diamond shape to illustrate the supersonic d'Alembert's Paradox.

It can be observed that only the area that "seems" to be by the flow was used in expressing equation (15.38). The relation between P_2 and P_4 is such that the flow depends on the upstream Mach number, M_1, and the specific heat, k. Regardless in the

equation of the state of the gas, the pressure at zone 2, P_2, is larger than the pressure at zone 4, P_4. Thus, there is always drag when the flow is supersonic which depends on the upstream Mach number, M_1, specific heat, k, and the "visible" area of the object. This drag is known in the literature as (shock) wave drag.

15.6 Flat Body with an Angle of Attack

Previously, the thickness of a body was shown to have a drag. Now, a body with zero thickness but with an angle of attack will be examined. As opposed to the thickness of the body, in addition to the drag, the body also obtains lift. Again, the slip condition is such that the pressure in region 5 and 7 are the same, and additionally the direction of the velocity must be the same. As before, the magnitude of the velocity will be different between the two regions.

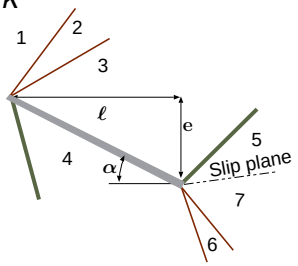


Fig. -15.8. The definition of attack angle for the Prandtl–Meyer function.

15.7 Examples For Prandtl–Meyer Function

Example 15.1:
A wall is included with 20.0° an inclination. A flow of air with a temperature of 20°C and a speed of $U = 450m/sec$ flows (see Figure 15.9). Calculate the pressure reduction ratio, and the Mach number after the bending point. If the air flows in an imaginary two–dimensional tunnel with width of $0.1[m]$ what will the width of this imaginary tunnel after the bend? Calculate the "fan" angle. Assume the specific heat ratio is $k = 1.4$.

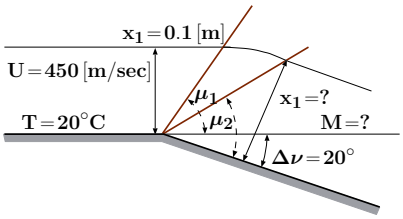


Fig. -15.9. Schematic for Example (14.5).

SOLUTION

First, the initial Mach number has to be calculated (the initial speed of sound).

$$a = \sqrt{k\,R\,T} = \sqrt{1.4 * 287 * 293} = 343.1m/sec$$

The Mach number is then

$$M = \frac{450}{343.1} = 1.31$$

this Mach number is associated with

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	μ
1.3100	6.4449	0.35603	0.74448	0.47822	52.6434

The "new" angle should be

$$\nu_2 = 6.4449 + 20 = 26.4449^\circ$$

and results in

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	μ
2.0024	26.4449	0.12734	0.55497	0.22944	63.4620

Note that $P_{01} = P_{02}$

$$\frac{P_2}{P_1} = \frac{P_{01}}{P_1}\frac{P_2}{P_{02}} = \frac{0.12734}{0.35603} = 0.35766$$

The "new" width can be calculated from the mass conservation equation.

$$\rho_1 x_1 M_1 c_1 = \rho_2 x_2 M_2 c_2 \Longrightarrow x_2 = x_1 \frac{\rho_1}{\rho_2}\frac{M_1}{M_2}\sqrt{\frac{T_1}{T_2}}$$

$$x_2 = 0.1 \times \frac{0.47822}{0.22944} \times \frac{1.31}{2.0024}\sqrt{\frac{0.74448}{0.55497}} = 0.1579[m]$$

Note that the compression "fan" stream lines are note and their function can be obtain either by numerical method of going over small angle increments. The other alternative is using the exact solution[1]. The expansion "fan" angle changes in the Mach angle between the two sides of the bend

$$\text{fan angle} = 63.4 + 20.0 - 52.6 = 30.8^\circ$$

End Solution

Reverse the example, and this time the pressure on both sides are given and the angle has to be obtained[2].

[1] It isn't really different from this explanation but shown in a more mathematical form, due to Landau and friends. It will be presented in the future version. It isn't present now because of the low priority to this issue.

[2] This example is for academic understanding. There is very little with practicality in this kind of problem.

Example 15.2:

Gas with $k = 1.67$ flows over bend (see Figure 15.2). The gas flow with Mach 1.4 and Pressure 1.2[Bar]. It is given that the pressure after the turning is 1[Bar]. Compute the Mach number after the bend, and the bend angle.

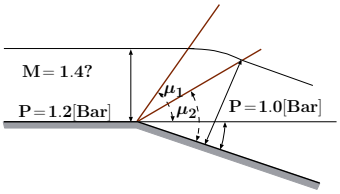


Fig. -15.10. Schematic for Example (14.5).

SOLUTION

The Mach number is determined by satisfying the condition that the pressure downstream and the Mach are given. The relative pressure downstream can be calculated by the relationship

$$\frac{P_2}{P_{02}} = \frac{P_2}{P_1}\frac{P_1}{P_{01}} = \frac{1}{1.2} \times 0.31424 = 0.2619$$

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	μ
1.4000	7.7720	0.28418	0.60365	0.47077	54.4623

With this pressure ratio $\bar{P} = 0.2619$ require either locking in the table or using the enclosed program.

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	μ
1.4576	9.1719	0.26190	0.58419	0.44831	55.5479

For the rest of the calculation the initial condition is used. The Mach number after the bend is $M = 1.4576$. It should be noted that specific heat isn't $k = 1.4$ but $k = 1.67$. The bend angle is

$$\Delta\nu = 9.1719 - 7.7720 \sim 1.4^\circ$$

$$\Delta\mu = 55.5479 - 54.4623 = 1.0^\circ$$

End Solution

15.8 Combination of the Oblique Shock and Isentropic Expansion

Example 15.3:

Consider two–dimensional flat thin plate at an angle of attack of 4° and a Mach number of 3.3. Assume that the specific heat ratio at stage is $k = 1.3$, calculate the drag coefficient and lift coefficient.

SOLUTION

For $M = 3.3$, the following table can be obtained:

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	μ
3.3000	62.3113	0.01506	0.37972	0.03965	73.1416

With the angle of attack the region 3 will be at $\nu \sim 62.31 + 4$ for which the following table can be obtained (Potto-GDC)

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	μ
3.4996	66.3100	0.01090	0.35248	0.03093	74.0528

On the other side, the oblique shock (assuming weak shock) results in

$\mathbf{M_x}$	$\mathbf{M_{y_s}}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{s}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.3000	0.43534	3.1115	88.9313	20.3467	4.0000	0.99676

and the additional information, by clicking on the minimal button, provides

$\mathbf{M_x}$	$\mathbf{M_{y_w}}$	$\theta_\mathbf{w}$	δ	$\frac{\mathbf{P_y}}{\mathbf{P_x}}$	$\frac{\mathbf{T_y}}{\mathbf{T_x}}$	$\frac{\mathbf{P_{0_y}}}{\mathbf{P_{0_x}}}$
3.3000	3.1115	20.3467	4.0000	1.1157	1.1066	0.99676

The pressure ratio at point 3 is

$$\frac{P_3}{P_1} = \frac{P_3}{P_{03}}\frac{P_{03}}{P_{01}}\frac{P_{01}}{P_1} = 0.0109 \times 1 \times \frac{1}{0.01506} \sim 0.7238$$

The pressure ratio at point 4 is

$$\frac{P_3}{P_1} = 1.1157$$

$$d_L = \frac{2}{kP_1{M_1}^2}(P_4 - P_3)\cos\alpha = \frac{2}{k{M_1}^2}\left(\frac{P_4}{P_1} - \frac{P_3}{P_1}\right)\cos\alpha$$

$$d_L = \frac{2}{1.33.3^2}\left(1.1157 - 0.7238\right)\cos 4^\circ \sim .054$$

$$d_d = \frac{2}{k{M_1}^2}\left(\frac{P_4}{P_1} - \frac{P_3}{P_1}\right)\sin\alpha = \frac{2}{1.33.3^2}\left(1.1157 - 0.7238\right)\sin 4^\circ \sim .0039$$

This shows that on the expense of a small drag, a large lift can be obtained. Discussion on the optimum design is left for the next versions.

End Solution

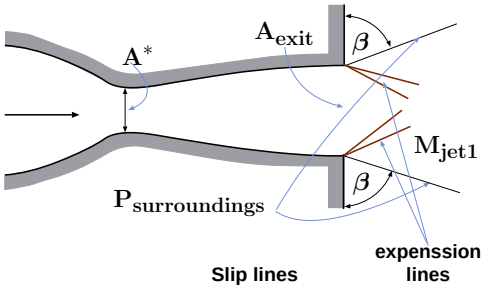


Fig. -15.11. Schematic of the nozzle and Prandtl–Meyer expansion.

Example 15.4:

To understand the flow after a nozzle consider a flow in a nozzle shown in Figure 15.4. The flow is choked and additionally the flow pressure reaches the nozzle exit above the surrounding pressure. Assume that there is an isentropic expansion (Prandtl–Meyer expansion) after the nozzle with slip lines in which there is a theoretical angle of expansion to match the surroundings pressure with the exit. The ratio of exit area to throat area ratio is 1:3. The stagnation pressure is 1000 [kPa]. The surroundings pressure is 100[kPa]. Assume that the specific heat, $k = 1.3$. Estimate the Mach number after the expansion.

SOLUTION

The Mach number a the nozzle exit can be calculated using Potto-GDC which provides

$\mathbf{M}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	$\frac{\mathbf{A}}{\mathbf{A^\star}}$	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{A}\times\mathbf{P}}{\mathbf{A^*}\times\mathbf{P_0}}$	$\frac{\mathbf{F}}{\mathbf{F^*}}$
1.7632	0.61661	0.29855	1.4000	0.18409	0.25773	0.57478

Thus the exit Mach number is 1.7632 and the pressure at the exit is

$$P_{exit} = P_0 \frac{P - exit}{P - 0} = 1000 \times 0.18409 = 184.09[kPa]$$

This pressure is higher than the surroundings pressure and additional expansion must occur. This pressure ratio is associated with a expansion angle that Potto-GDC provide as

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_0}$	μ
1.7632	19.6578	0.18409	0.61661	0.29855	60.4403

The need additional pressure ratio reduction is

$$\frac{P_{surroundings}}{P_0} = \frac{P_{surroundings}}{P_{exit}} \frac{P_{exit}}{P_0} = \frac{100}{184.09} \times 0.18409 = 0.1$$

Potto-GDC provides for this pressure ratio

$\mathbf{M}$	ν	$\frac{\mathbf{P}}{\mathbf{P_0}}$	$\frac{\mathbf{T}}{\mathbf{T_0}}$	$\frac{\rho}{\rho_\mathbf{0}}$	μ
2.1572	30.6147	0.10000	0.51795	0.19307	65.1292

The change of the angle is

$$\Delta\text{angle} = 30.6147 - 19.6578 = 10.9569$$

Thus the angle, β is

$$\beta = 90 - 10.9569 \sim 79.0$$

The pressure at this point is as the surroundings. However, the stagnation pressure is the same as originally was enter the nozzle! This stagnation pressure has to go through serious of oblique shocks and Prandtl-Meyer expansion to match the surroundings stagnation pressure.

End Solution

APPENDIX A

Computer Program

A.1 About the Program

The program is written in a C++ language. This program was used to generate all the data in this book. Some parts of the code are in FORTRAN (old code especially for chapters 12 and 13 and not included here.[1]. The program has the base class of basic fluid mechanics and utilities functions to calculate certain properties given data. The derived class are Fanno, isothermal, shock and others.

At this stage only the source code of the program is available no binary available. This program is complied under gnu g++ in /Gnu/Linux system. As much support as possible will be provided if it is in Linux systems. NO Support whatsoever will be provided for any Microsoft system. In fact even PLEASE do not even try to use this program under any Microsoft window system.

A.2 Usage

To use the program some information has to be provided. The necessary input parameter(s), the kind of the information needed, where it has to be in a LaTeX format or not,

[1] when will be written in C++ will be add to this program.

and in many case where it is a range of parameter(s). **machV** The Mach number and it is used in

fldV The $\frac{4fL}{D}$ and it is used in Fanno class is

p2p1V The pressure ratio of the two sides of

M1V Entrance Mach M1 to the tube Fanno

M1ShockV Entrance Mach M1 when expec Fanno and isothermal classes

FLDShockV FLD with shock in the in Fanno

M1fldV both M_1 and $\frac{4fL}{D}$ are given

M1fldP2P1V three part info $\frac{P_1}{P_2}$, M_1 and $\frac{4f}{L}$

MxV M_x or M_y

infoStagnation print standard (stagnation) info

infoStandard standard info for (Fanno, shock etc)

infoTube print tube side info for (Fanno, etc) including

infoShock print shock sides info

infoTubeShock print tube info shock main info

infoTubeProfile the Mach number and pressure ratio profiles

infoTubeShockLimits print tube limits with shock

To get the shock results in LaTeX of M_x The following lines have to be inserted in the end of the main function.

```
int isTex = yes;
int isRange = no;
whatInfo = infoStandard ;
variableName =  MxV;
Mx = 2.0 ;
s.makeTable(whatInfo, isRange, isTex,  variableName, variableValue);

****************************************
The following stuff is the same as above/below
if you use showResults with showHeads but the
information is setup for the latex text processing.
You can just can cut and paste it in your latex file.
You must use longtable style file and dcolumn
style files.
```

```
*****************************************
\setlongtables
\begin{longtable}
{|D..{1.4}|D..{1.4}|D..{1.4}|D..{1.4}|D..{1.4}|D..{1.4}|D..{1.4}|}
\caption{ ??  \label{?:tab:?}}\\
\hline
\multicolumn{1}{|c|} {$\rule[-0.1in]{0.pt}{0.3 in}\mathbf{M} $}  &
\multicolumn{1}{|c|} {$\mathbf{4fL \over D} $}  &
\multicolumn{1}{|c|} {$\mathbf{P \over P^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{P_0 \over {P_0}^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{\rho \over \rho^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{U \over {U}^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{T \over T^{*}} $}

\\\hline

\endfirsthead
\caption{ ?? (continue)} \\\hline
\multicolumn{1}{|c|} {$\rule[-0.1in]{0.pt}{0.3 in}\mathbf{M} $}  &
\multicolumn{1}{|c|} {$\mathbf{4fL \over D} $}  &
\multicolumn{1}{|c|} {$\mathbf{P \over P^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{P_0 \over {P_0}^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{\rho \over \rho^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{U \over {U}^{*}} $}  &
\multicolumn{1}{|c|} {$\mathbf{T \over T^{*}} $}

\\\hline
\endhead
    2.176&    2.152&   0.3608&    1.000&   0.5854&    3.773&   0.6164 \\
\hline\end{longtable}
```

A.3 Program listings

Can be download from www.potto.org.

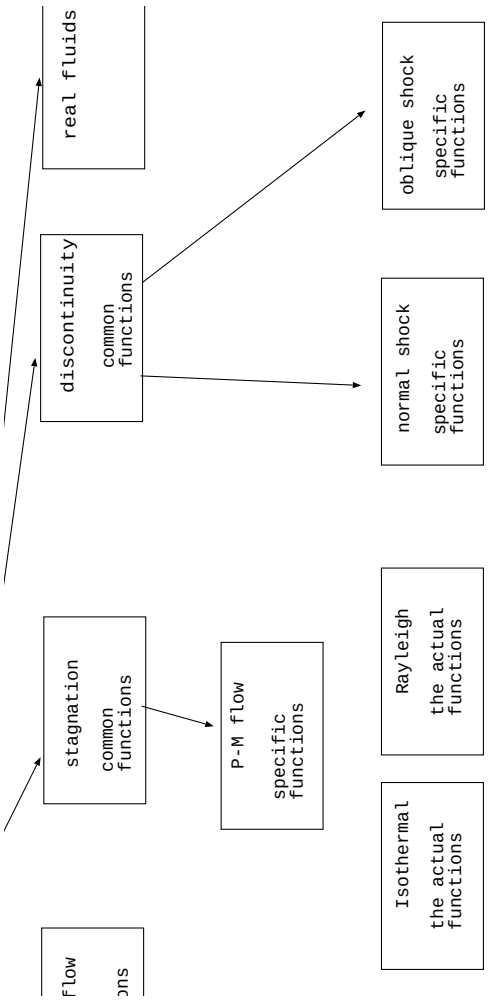


Fig. -A.1. Schematic diagram that explains the structure of the program

APPENDIX B

Oblique Shock History

The oblique shock is dealing with a physical phenomenon of the flow of compressible substance above supersonic velocity over positive inclined plane. When the incline angle is negative (the plane turn away from the flow) the flow turn around the corner without a shock (Prandtl-Meyer Flow). The question when the flow undergoes a oblique shock or when the Prandtl–Meyer flow occur was never really settled. Additionally, when the incline plane angle increase above a certain value the oblique shock turns into detached shock (different kind shock). The detached shock is a shock that does not touch the body (which affects the resistance). While the value was known, there was no mathematical explanations for it.

During the World War Two, there was importance to having airplane flying faster than the speed of sound which is physical barrier that was cause by the shock. There are several kinds of shocks and some view the most important as the oblique shock. Consequently, many governments and agencies pour money to delve this phenomenon.

NASA sponsor researched into this problems and produced a famous report known as NACA 1135 which declared that the oblique shock's problem no analytical solution can be obtained. Since this challenge declaration was put into the open, countless people have attempted to solve it. And in the tradition of compressible flow, everything of significant has to be discovered several times before it accepted as "discovered." The first one to discover the analytical solution was Briggs, J. after 8 years after the challenge was issued. Six years later, for the second time it was rediscovered by Mascitti, V.R. The emergence of the new mathematical approximation tools had "buried" these two solutions for the next thirty years. And countless thesis's using small perturbation, artificial viscosity, etc were implemented to solve this problem. However, the "hand waving" presentation in classes daunted the scientific community and the solution was rediscovered again by Wolf, T., (1993). And of course that was

not enough it has to rediscovered by George Emanuel, (2000).

At this stage, the information about the discovery had some penetration into the field and it started to appear in another book Anderson's book. However, examination of many books since that date shows that the rest of the authors were not aware of the analytical solution. For example, every popular books published in this millennium show no reference to any of analytical solutions. In fact, one can not be sure that no another analytical solution published somewhere else. It is interesting that the authors of all these variations solutions believed and emphasize the usage of the analytical expression to obtain the numerical value of the solution. None of them saw the real importance of the studies the limits of the solution or when it is applicable.

Again, in early 2003 Bar-Meir rediscovered the solution. However, there was a twist to the way the solution was presented. As oppose to the previous authors presented their solution only in academic publication, this time the solution was published in an open content publication. In this open content publication, a dialog between the author and his readership was very strong as opposite to "committee" (peer reviewers) that approved all the other publications. The difference between the two publication methods turned out to be astounding. Numerous people attacked, criticized, scrutinized, and enhanced and the solution (even the author). As result, the solution was enhanced from version 0.3x of the book to current version 0.4.4.2

The difference of Bar-Meir's solution to all the previous solutions is not that there is any different numerical solutions. This difference is in the importance of better understanding of the physical phenomenon and its limits. Now the boundaries of the oblique shock model can be explained without resulting into "hand waving." For example, the detach shock wave can be explained why there is detached shock with having, the students scratching their heads. More importunately something that was a puzzle before can be explained. The oblique shock occurs when the inclination is possible.However, what happened when the inclination plane is zero or even negative. No one really solved this issue even though numerous works were carried assuming that there is a solution. Bar-Meir's solution was able to demonstrate that no oblique shock can occur when the inclination is zero, by default voiding the significance the numerous Ph.D's thesis.

The interesting the part of Bar-Meir's solution is that was never published in any scientific paper and yet was read by numerous practitioners and students in the field. Thus, rediscovering the analytical solution to oblique shock will be now like rediscovering the calculus. The power of the distribution of the information via the open content is much faster and wider than the regular academic publication system. Another important part of the way the material was published is who learn about it first. In the tradition publication the establishment like old professors who are acting as the reviewers learn about new information first. Later, they pass this information to the rest of the community especially to the students. In the new system, the students and many practitioners in the field know about it long before the establishment even aware that a new idea has brought to forefront.

Thus, the open content process moves the power of knowledge into the mass and remove the continuous rediscovering, so that improving and enhancing the solution

be carried out right way.

1. Briggs, J. "Comment on Calculation of Oblique shock waves," AIAA Journal Vol 2, No 5 p. 974, 1963.

2. Mascitti, V.R. "A close-Form Solution to Oblique Shock-Wave Properties," J. Aircraft 6, 66 (1969) "A close-Form Solution to Oblique Shock-Wave

3. Wolf, T., "Comment on 'Approximate Formula of Weak Oblique shock wave angle," AIAA J. 31, 1363 (1993).

4. George Emanuel, analytical fluid dynamics, crc press (2000)

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